Optimal Antennas: Operators, Limits, and Design — Short Course
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1 Representation

The surface current density $\boldsymbol{J}$ is expressed in terms of basis functions $\boldsymbol{\psi}_n$ as

$$\boldsymbol{J}(\boldsymbol{r}) \approx \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n(\boldsymbol{r}), \tag{1}$$

so that the power/energy quadratic forms are transformed from their analytical prescription to the algebraic form

$$p = \frac{1}{2}\int_{\Omega} \boldsymbol{J}^*(\boldsymbol{r}) \cdot \mathcal{L}\left\{\boldsymbol{J}(\boldsymbol{r}')\right\} \mathrm{d}\boldsymbol{r} \quad \longrightarrow \quad p \approx \frac{1}{2}\mathbf{I}^{\mathrm{H}}\mathbf{L}\mathbf{I} \quad \text{with} \quad L_{mn} \equiv \int_{\Omega} \boldsymbol{\psi}_m(\boldsymbol{r}) \cdot \mathcal{L}\left\{\boldsymbol{\psi}_n(\boldsymbol{r}')\right\} \mathrm{d}\boldsymbol{r}. \tag{2}$$

2 Method of moments

$$\mathbf{Z}\mathbf{I} = (\mathbf{R}_0 + \mathbf{R}_\rho + \mathrm{j}\mathbf{X}_0 + \mathrm{j}\mathbf{X}_\rho)\,\mathbf{I} = \mathbf{V} \tag{3}$$

$$\mathbf{Z}_0 = \mathbf{R}_0 + \mathrm{j}\mathbf{X}_0 \tag{4}$$

$$R_{\rho,mn} = \int_{\Omega} R_{\mathrm{s}}(\boldsymbol{r})\,\boldsymbol{\psi}_m(\boldsymbol{r}) \cdot \boldsymbol{\psi}_n(\boldsymbol{r})\,\mathrm{d}S, \tag{5}$$

$$V_n = \int_{\Omega} \boldsymbol{\psi}_n(\boldsymbol{r}) \cdot \boldsymbol{E}_{\mathrm{i}}(\boldsymbol{r})\,\mathrm{d}S. \tag{6}$$

$$F_n(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}) = \frac{-\mathrm{j}Z_0 k}{4\pi}\int_{\Omega} \hat{\boldsymbol{e}} \cdot \boldsymbol{\psi}_n(\boldsymbol{r})\,\mathrm{e}^{\mathrm{j}k\hat{\boldsymbol{r}}\cdot\boldsymbol{r}}\,\mathrm{d}S \tag{7}$$

3 Related matrix operators

Stored energy matrix $\mathbf{W}$ is defined as

$$\mathbf{W} = \operatorname{Im}\left\{\omega \frac{\partial \mathbf{Z}_0}{\partial \omega}\right\}, \tag{8}$$

where

$$\mathbf{X}_{\mathrm{m}} = \frac{1}{2}(\mathbf{W} + \mathbf{X}_0), \tag{9}$$

$$\mathbf{X}_{\mathrm{e}} = \frac{1}{2}(\mathbf{W} - \mathbf{X}_0). \tag{10}$$

4 Power quantities

5 Antenna parameters

- Antenna directivity for direction $\hat{\boldsymbol{e}}$ and polarization $\hat{\boldsymbol{e}}$:

$$D\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}\right) = \frac{2\pi}{Z_0} \frac{\left|\mathbf{F}\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}\right)\mathbf{I}\right|^2}{\mathbf{I}^{\mathrm{H}}\mathbf{R}_0\mathbf{I}} \tag{11}$$

- Antenna gain:

$$G\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}\right) = \frac{2\pi}{Z_0} \frac{\left|\mathbf{F}\left(\hat{\boldsymbol{r}}, \hat{\boldsymbol{e}}\right)\mathbf{I}\right|^2}{\mathbf{I}^{\mathrm{H}}\left(\mathbf{R}_0 + \mathbf{R}_\rho\right)\mathbf{I}} \tag{12}$$

- Antenna (radiation) Q-factor:

$$Q = \frac{\max\left\{\mathbf{I}^{\mathrm{H}}\mathbf{X}_{\mathrm{m}}\mathbf{I}, \mathbf{I}^{\mathrm{H}}\mathbf{X}_{\mathrm{e}}\mathbf{I}\right\}}{\mathbf{I}^{\mathrm{H}}\mathbf{R}_0\mathbf{I}} = \frac{1}{2}\frac{\mathbf{I}^{\mathrm{H}}\mathbf{W}\mathbf{I}}{\mathbf{I}^{\mathrm{H}}\mathbf{R}_0\mathbf{I}} + \frac{\left|\mathbf{I}^{\mathrm{H}}\mathbf{X}_0\mathbf{I}\right|}{\mathbf{I}^{\mathrm{H}}\mathbf{R}_0\mathbf{I}} \tag{13}$$

- Radiation efficiency:

$$\eta_{\mathrm{rad}} = \frac{\mathbf{I}^{\mathrm{H}}\mathbf{R}_0\mathbf{I}}{\mathbf{I}^{\mathrm{H}}\left(\mathbf{R}_0 + \mathbf{R}_\rho\right)\mathbf{I}} \tag{14}$$

6 QCQP and Its Solution

QCQP for two constraints with matrices $\mathbf{A}, \mathbf{C} \succ \mathbf{0}$

$$\begin{aligned} \underset{\mathbf{I}}{\text{minimize}} \quad & \mathbf{I}^{\mathrm{H}}\mathbf{A}\mathbf{I} \\ \text{subject to} \quad & \mathbf{I}^{\mathrm{H}}\mathbf{B}\mathbf{I} = 0 \\ & \mathbf{I}^{\mathrm{H}}\mathbf{C}\mathbf{I} = 1, \end{aligned} \tag{15}$$

can be solved via parameterized eigenvalue problem as

$$\left(\mathbf{A} + \nu\mathbf{B}\right)\mathbf{I}_i = \lambda_i\mathbf{C}\mathbf{I}_i, \tag{16}$$

maximizing the value of the lowest eigenvalue $\lambda_1 = \min_i \lambda_i\left(\nu\right)$.

7 MoM Matrices

The vacuum impedance matrix $\mathbf{Z}_0$ and its derivative are defined by the following blocks:

$$\mathbf{Z}_0 = \mathbf{R}_0 + \mathrm{j}\mathbf{X}_0 = \mathrm{j}Z_0 a^2\left(ka\mathbf{Z}_0^{\mathrm{m}} - \frac{1}{ka}\mathbf{Z}_0^{\mathrm{e}}\right), \tag{17}$$

$$\omega\frac{\partial\mathbf{Z}_0}{\partial\omega} = \mathrm{j}Z_0 a^2\left(ka\left(\mathbf{Z}_0^{\mathrm{m}} - \mathrm{j}ka\mathbf{A}^{\mathrm{m}}\right) + \frac{1}{ka}\left(\mathbf{Z}_0^{\mathrm{e}} + \mathrm{j}ka\mathbf{A}^{\mathrm{e}}\right)\right), \tag{18}$$

where the particular matrices are implemented as

$$Z_{0,mn}^{\mathrm{m}} = \frac{1}{a^3}\int\limits_{\Omega}\int\limits_{\Omega}\boldsymbol{\psi}_m\left(\boldsymbol{r}\right)\cdot\boldsymbol{\psi}_n\left(\boldsymbol{r}'\right)\frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi R}\,\mathrm{d}S\,\mathrm{d}S', \tag{19}$$

$$Z_{0,mn}^{\mathrm{e}} = \frac{1}{a}\int\limits_{\Omega}\int\limits_{\Omega}\nabla\cdot\boldsymbol{\psi}_m\left(\boldsymbol{r}\right)\nabla'\cdot\boldsymbol{\psi}_n\left(\boldsymbol{r}'\right)\frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi R}\,\mathrm{d}S\,\mathrm{d}S', \tag{20}$$

$$A_{mn}^{\mathrm{m}} = \frac{1}{a^4}\int\limits_{\Omega}\int\limits_{\Omega}\boldsymbol{\psi}_m\left(\boldsymbol{r}\right)\cdot\boldsymbol{\psi}_n\left(\boldsymbol{r}'\right)\frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi}\,\mathrm{d}S\,\mathrm{d}S', \tag{21}$$

$$A_{mn}^{\mathrm{e}} = \frac{1}{a^2}\int\limits_{\Omega}\int\limits_{\Omega}\nabla\cdot\boldsymbol{\psi}_m\left(\boldsymbol{r}\right)\nabla'\cdot\boldsymbol{\psi}_n\left(\boldsymbol{r}'\right)\frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi}\,\mathrm{d}S\,\mathrm{d}S'. \tag{22}$$