Pareto Optimality Between Prescribed Far-Field Pattern and Radiation Efficiency

(Very) Preliminary Results

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Workshop
Les Marécottes
Outline

1. Current Density Bounds
2. Far-field Optimality
3. Methodology
4. Example #1
5. Compact Representation of the Far Field
6. Realization of “Arbitrary” Far Field
7. Example #2
8. Unknown Phase
9. Concluding Remarks
Czech Technical University in Prague

Established in 1707 as the first non-military technical university in Europe.
- From 12 students in 1707 to more than 20,000 students around 2020.

Left: Prague; right: CTU, Faculty of Electrical Engineering (one of eight faculties).

You are welcome to visit us in Prague!
Current Density Bounds

- Draw whatever current you want to extremize a given metric $f(I)$.

\[
\begin{align*}
\text{minimize} & \quad f(I) \\
\text{subject to} & \quad g_i(I) \leq c_i
\end{align*}
\]

- Typically QCQP (or SDP).
- Full quadratic forms . . .
- Substructures, port modes, . . .
Current Density Bounds

- Draw whatever current you want to extremize a given metric \( f(I) \).

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\begin{align*}
\text{minimize} & \quad \mathbf{I}^H \mathbf{A}_i \mathbf{I} \\
\text{subject to} & \quad \mathbf{I}^H \mathbf{B}_i \mathbf{I} \leq c_i
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- Typically QCQP (or SDP).
- Full quadratic forms …
- Substructures, port modes, …
Current Density Bounds

- Advanced for many scalar metrics, e.g.,
  - Q-factor\(^1\) (bandwidth),
  - gain\(^2\),
  - scattering\(^3\),
  - optics\(^4\),
  - realized gain\(^5\),
  - trade-offs\(^6\),
  - ...


A Note: MoM Solution × Current Impressed in Vacuum

MoM solution

Solution to $\mathbf{I} = \mathbf{Z}^{-1}\mathbf{V}$ for an incident plane wave.
A Note: MoM Solution $\times$ Current Impressed in Vacuum

MoM solution

Solution to $\mathbf{I} = \mathbf{Z}^{-1}\mathbf{V}$ for an incident plane wave.

Current impressed in vacuum

Solution to $\mathbf{X}\mathbf{I}_i = \lambda_i\mathbf{R}\mathbf{I}_i$ (the first inductive mode).

- Looking for an optimal current, it can be chosen completely freely, only the excitation $\mathbf{V} = \mathbf{Z}\mathbf{I}$ may not be realizable.
How to deal with far-field optimality?

- Point-wise, \textit{i.e.}, directivity $D (\hat{e}, \hat{r})$ or gain $G (\hat{e}, \hat{r})$.
- Prescribed far-field $F = F (\vartheta, \varphi)$:
  - is a vector function,
  - with a (unknown) phase,
  - required smoothness ($F (\vartheta, \varphi) \rightarrow F (\vartheta_p, \varphi_p)$).
Far-field Optimality

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Some works exist

- For example, for the cost in Q-factor\(^7\),
- far-field shaping for small antennas (SDP)\(^8\),
- ...

---


Far-field Optimality

\[ I = ? \]

MoM solution

plate
The hypothesis

“Almost every far field pattern $F_0$ can be generated by a current $I_0$, however, potentially at the cost of almost zero radiation efficiency.”

\[
\begin{align*}
\text{minimize} \quad & \varepsilon_F = \|F_0 - F(I)\| \\
\text{subject to} \quad & \eta_{\text{rad}}(I) \leq x
\end{align*}
\]

- The problem above forms a Pareto frontier in $\varepsilon_F(I)$ and $\eta_{\text{rad}}(I)$.
- A type of norm taken $| \cdot |$ is crucial.
Far-field Optimality: A Role of Phase

Far field $F_0$ has to be given to solve

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\end{align*}$$

for a given $x$. 
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Two possibilities differing in our knowledge of the desired far field $F_0$:
Far-field Optimality: A Role of Phase

Far field $F_0$ has to be given to solve

$$\min_{\mathbf{I}} \quad \varepsilon_F = \|\mathbf{F}_0 - \mathbf{F}(\mathbf{I})\|$$

subject to \( \eta_{\text{rad}}(\mathbf{I}) \leq x \)

for a given \( x \).

Two posibilities differing in our knowledge of the desired far field $F_0$:

**Problem #1:**

- Both amplitude and phase of $F_0$:

  $$\min_{\mathbf{I}} \quad |\mathbf{F}_0 - \mathbf{F}(\mathbf{I})|^2$$

  subject to \( \eta_{\text{rad}}(\mathbf{I}) \leq x \)

- Phase often arbitrary.
Far-field Optimality: A Role of Phase

Far field $F_0$ has to be given to solve

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for a given $x$.

Two possibilities differing in our knowledge of the desired far field $F_0$:

**Problem #1:**

- Both amplitude and phase of $F_0$:

$$\min_I \quad |F_0 - F(I)|^2$$

subject to $\eta_{\text{rad}}(I) \leq x$

- Phase often arbitrary.

**Problem #2:**

- Amplitude of $F_0$ is known; phase is arbitrary:

$$\min_I \quad \| |F_0| - |F(I)| \|^2$$

subject to $\eta_{\text{rad}}(I) \leq x$

- Hard to solve ($\propto \text{MAX-CUT} \rightarrow \text{NP-hard}$).
Methodology – Operators

- RWG representation of MoM IE operators.

\[
F(\hat{\mathbf{r}}) = F(\hat{\mathbf{\vartheta}}, \hat{\mathbf{r}}),
\]

\[
F(\hat{\mathbf{e}}, \hat{\mathbf{r}}) = K(\hat{\mathbf{e}}, \hat{\mathbf{r}}) I,
\]

with \( K = [K_p] \) point-wise given as

\[
K_p(\hat{\mathbf{e}}, \hat{\mathbf{r}}) = -j\frac{Z_0}{4\pi} k^3 \hat{\mathbf{e}} \cdot \psi_p(r_1) e^{jk\hat{\mathbf{r}} \cdot r_1} dr_1.
\]

- Impedance matrix \( Z = R + L + jX \) with \( R + jX \) being a vacuum part, and \( L \) representing ohmic losses, point-wise as

\[
L_{pq} = Z_{\Omega} R_s(\mathbf{r}) \psi_p(\mathbf{r}) \cdot \psi_q(\mathbf{r}) d\Omega.
\]
Methodology – Operators

- RWG representation of MoM IE operators.
- Far field

\[ F(\hat{r}) = \begin{bmatrix} F(\hat{\vartheta}, \hat{r}) \\ F(\hat{\varphi}, \hat{r}) \end{bmatrix}. \]
Methodology – Operators

- RWG representation of MoM IE operators.
- Far field

\[ F(\hat{r}) = \begin{bmatrix} F(\hat{\vartheta}, \hat{r}) \\ F(\hat{\varphi}, \hat{r}) \end{bmatrix} \]

- Far field component \( F(\hat{e}, \hat{r}) = K(\hat{e}, \hat{r}) I \), with \( K = [K_p] \) point-wise given as

\[ K_p(\hat{e}, \hat{r}) = -j\frac{Z_0 k}{4\pi} \int_{\mathbb{R}^3} \hat{e} \cdot \psi_p(r_1) e^{jk \hat{r} \cdot r_1} \, dV_1. \]
Methodology – Operators

- RWG representation of MoM IE operators.
- Far field

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with

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- Impedance matrix

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Z = R + L + jX
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with \(R + jX\) being a vacuum part, and \(L\) representing ohmic losses, point-wise as

\[
L_{pq} = \int_\Omega R_s(r) \psi_p^*(r) \cdot \psi_q(r) \ d\Omega
\]

(e.g., thin-sheet model).
Methodology – Antenna Metrics

- **Radiation efficiency**

\[
\eta_{\text{rad}} = \frac{P_{\text{rad}}}{P_{\text{rad}} + P_{\text{lost}}} \approx \frac{I^HRI}{I^H(R + L)I} = \frac{1}{1 + \delta}
\]

with \( \delta = P_{\text{lost}}/P_{\text{rad}} \) being dissipation factor.

- **Far field**

\[
F(\hat{e}, \hat{r}) \approx K(\hat{e}, \hat{r})I.
\]

- **Antenna gain**

\[
G(\hat{e}, \hat{r}) = \frac{2\pi |F(\hat{e}, \hat{r})|^2}{Z_0 P_{\text{rad}} + P_{\text{lost}}} \approx \frac{4\pi I^H K^H(\hat{e}, \hat{r}) K(\hat{e}, \hat{r})I}{Z_0 I^H(R + L)I}.
\]
Methodology – Far-Field Integration

- Radiation power

\[
P_{\text{rad}} = \frac{1}{2Z_0} \int_{4\pi} \mathbf{F}^* (\hat{r}) \cdot \mathbf{F} (\hat{r}) \, d\Omega \approx \frac{1}{2} I^H R I.
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Methodology – Far-Field Integration

- Radiation power

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- Lebedev quadrature over unit ball

\[ I = \int f(\Omega) \, d\Omega \approx \sum_n \Lambda_n f(\vartheta_n, \varphi_n) \]

\[ P_{\text{rad}} \approx \frac{1}{2Z_0} I^H [K]^H \Lambda [K] I. \]

with

\[ [K] = [K^T(\vartheta_1, \varphi_1) \ldots K^T(\vartheta_n, \varphi_N)]^T. \]
Methodology – Far-Field Integration

- Radiation power

\[ P_{\text{rad}} = \frac{1}{2Z_0} \int \frac{\mathbf{F}^*(\hat{\mathbf{r}}) \cdot \mathbf{F}(\hat{\mathbf{r}})}{4\pi} \, d\Omega \approx \frac{1}{2} I^H R I. \]

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with

\[ [\mathbf{K}] = [\mathbf{K}^T(\vartheta_1, \varphi_1) \ldots \mathbf{K}^T(\vartheta_n, \varphi_N)]^T. \]

- Analogy to guassian quadrature on spherical shell.

- Selected quadrature degree treats spherical harmonics exactly up to known order.
Let us focus on the **Problem #1** first. (Phase of $F_0$ is specified).
Problem #1 in RWG Basis

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\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2Z_0} | \Lambda^{1/2} (F_0 - [K]I) |^2 \\
\text{subject to} & \quad \frac{1}{2} I^H L I = \delta \\
& \quad \frac{1}{2} I^H R I = 1
\end{align*}
\]

- Quadratic program with two quadratic constraints.
- Relatively complicated optimized metric.
- The problem can rewritten preserving its original nature...
Problem #1 in RWG Basis – Simplification (Part 1)

Optimized metric is to be simplified

\[
\frac{1}{2Z_0} (F_0^H - I^H[K]^H) \Lambda (F_0 - [K]I) = \frac{1}{2Z_0} F_0^H \Lambda F_0 - \frac{1}{Z_0} \text{Re} (I^H[K]^H \Lambda F_0) + \frac{1}{2} I^H R I.
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\]

Normalization of far field \( F_0 \)

Let us assume for the rest of the talk that the desired far field is normalized so that

\[
P_{\text{rad,0}} \approx \frac{1}{2Z_0} F_0^H \Lambda F_0 = 1
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Envelope correlation coefficient

\[
E(F, F_0) = |\rho(F, F_0)|^2 = \left| \frac{I^HRI_0}{\sqrt{I_0^HRI_0I^HRI}} \right|^2 = \cdots = |I^HRI_0|^2
\]
Problem #1 in RWG Basis – Simplification (Part 2)

\[
\begin{align*}
\text{minimize} & \quad I \left( 2 - \frac{1}{Z_0} \text{Re} \left( I^H [K]^H \Lambda F_0 \right) \right) \\
\text{subject to} & \quad \frac{1}{2} I^H L I = \delta \\
& \quad \frac{1}{2} I^H R I = 1
\end{align*}
\]

▶ In MIMO, ECC is usually minimized, here we want to maximize!
▶ A possibility to reduce to QCQP with one quadratic constraint only...

Grouping \( R \) and \( L \) constraints and changing multipliers.
Problem #1 in RWG Basis – Simplification (Part 2)

minimize \[ \int \left( 2 - \frac{1}{Z_0} \text{Re} \left( I^H [K]^H \Lambda F_0 \right) \right) \]
subject to \[ \frac{1}{2} I^H LI = \delta \]
\[ \frac{1}{2} I^H RI = 1 \]

or equivalently

maximize \[ \int \frac{1}{Z_0} \text{Re} \left( I^H [K]^H \Lambda F_0 \right) = \text{Re}(\rho(F, F_0)) \]
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► In MIMO, ECC is usually minimized, here we want to maximize!
► A possibility to reduce to QCQP with one quadratic constraint only...  
  ► Grouping R and L constraints and changing multipliers.
Two parallel plates, $k\ell = \pi$, copper $\sigma = 5.96 \cdot 10^7 \text{Sm}^{-1}$.

- Two plates shown above are used everywhere in this talk as an example.
Example #1: Synthesis of MoM Current

- Current $I_0$ is evaluated for an impinging plane wave (normal incidence, $\hat{z}$ polarization).
Example #1: Synthesis of MoM Current

- Current $I_0$ is evaluated for an impinging plane wave (normal incidence, $\hat{\mathbf{x}}$ polarization).
- Desired far field is specified as $F_0 = \Lambda^{1/2}[K]I_0$, $k\ell = \pi/4$, Lebedev quadrature of degree 50 ($L_{\text{max}} = 5$).
Example #1: Synthesis of MoM Current

- Current $\mathbf{I}_0$ is evaluated for an impinging plane wave (normal incidence, $\hat{x}$ polarization).
- Desired far field is specified as $\mathbf{F}_0 = \Lambda^{1/2} [\mathbf{K}] \mathbf{I}_0$, $k\ell = \pi/4$, Lebedev quadrature of degree 50 ($L_{\text{max}} = 5$).
- $\eta_{\text{rad,0}} \approx 0.9998$
Example #1

MoM solution: $\delta_0 = 4 \cdot 10^3$

$E(F, F_0)$

$\eta_{rad}$

$|F_0 - F|^2 / (2Z_0)$

Duality gap $g_N$

$g_N = \frac{|p - d|}{|p + d|}$
Example #1

Optimal current for point A \( (\delta \approx 10^{-4}) \).
Optimal current for point B \( (\delta \approx 3.16 \times 10^{-4}) \).
Optimal current for point C \( (\delta \approx 10^{-3}) \).

MoM solution: \( \delta_0 = 4 \times 10^3 \)

duality gap, solve:
\[
\lambda_{\text{min}} R_{I_n} = \Lambda_{\text{first}}
\]

\( E(F, F_0) \)
\( \eta_{\text{rad}} \)
\( |F_0 - F|^2 / (2Z_0) \)
duality gap \( g_N \)

optimized quantities
Example #1

Optimal current for point A (δ ≈ 10^{-4}).

Optimal current for point B (δ ≈ 3.16 · 10^{-4}).

Optimal current for point C (δ ≈ 10^{-3}).

Mathematical equations:

\[ E(F, F_0) \]

\[ \eta_{\text{rad}} \]

\[ |F_0 - F|^2 / (2Z_0) \]

Duality gap \( g_N \)

Optimized quantities
Optimal current for point **A**  
(\( \delta \approx 10^{-4} \)).

Optimal current for point **B**  
(\( \delta \approx 3.16 \cdot 10^{-4} \)).

Optimal current for point **C**  
(\( \delta \approx 10^{-3} \)).
Comparison of Far Fields $F_0$ and $F$

Desired far field $F_0$. 

Synthesized far field points B and C.
Comparison of Far Fields $F_0$ and $F$

Desired far field $F_0$.

Synthetized far field $F$ points $B$ and $C$. 
Entire-domain Basis For Compact Far-Field Representation

- The solution is constructed from many degrees of freedom (as many as basis functions).
- No possibility to further restrict the solution.
- No relationship to excitation possibilities.
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**Lossy Characteristic Modes**

\[ \Xi I_n = \lambda_n (R + L) I_n \]

- Potentially sparse basis of entire-domain functions.
- Relation to Lebedev quadrature and required number of points.
- Different properties than the classical characteristic modes \( \Xi I_n = \lambda_n R I_n \).
Power terms $P_{rad,mn} = \frac{I_m^H R I_n}{2}$ (left) and $P_{lost,mn} = \frac{I_m^H L I_n}{2}$ (right).

Two rectangular plates $\ell \times \ell/2$, separated by $\ell/4$, made of copper, $k\ell = \pi$. 

$\eta_{rad,n} \geq 0.2$ and $\eta_{rad,n} < 0.2$
Maximum Gain as Inherent Property of LCMs

It can be shown\(^9\) that LCMs follows:

\[
G_{ub} (\hat{e}, \hat{r}) = \sum_n G_n (\hat{e}, \hat{r})
\]

![Graph showing the relationship between maximum gain and angular frequency for different modes.]

- Curiously enough, the property above was unknown to both Harrington and Garbacz!

Optimal excitation coefficient:

\[ \beta_n = \sqrt{\frac{4\pi}{Z_0 G_{ub} (\hat{e}, \hat{r})} F_n^* (\hat{e}, \hat{r})} \]
Additional Insight for a Designer: Modal Parameters

\[ |\beta_n| \]

\[ \eta_{\text{rad},n} \geq 0.2 \quad \eta_{\text{rad},n} < 0.2 \]

Mode \( n \)
Additional Insight for a Designer: Modal Parameters

Modal significance:

\[
|t_n| = \left| \frac{1}{1 + j\lambda_n} \right|
\]
Additional Insight for a Designer: Modal Parameters

\[ |\beta_n| \]

\[ |t_n| \]

\[ \eta_{\text{rad}, n} \geq 0.2 \]
\[ \eta_{\text{rad}, n} < 0.2 \]
Additional Insight for a Designer: Modal Parameters

Modal radiation efficiency:

\[ \eta_{rad,n} = \frac{P_{rad,nn}}{P_{rad,nn} + P_{lost,nn}} = I_n^H R I_n \]
Additional Insight for a Designer: Modal Parameters

\[ |\beta_n|, |\eta_{rad,n}| \leq 0.2 \]
\[ |\beta_n|, |\eta_{rad,n}| < 0.2 \]

\[ |t_n|, \eta_{rad,n} \geq 0.2 \]
\[ |t_n|, \eta_{rad,n} < 0.2 \]
Additional Insight for a Designer: Modal Parameters

Modal Q-factor:

\[ Q_n = \frac{2\omega \max \{W_{m,n}, W_{e,n}\}}{P_{\text{rad}}} \]
Additional Insight for a Designer: Modal Parameters

\[ \eta_{rad,n} \geq 0.2 \quad \eta_{rad,n} < 0.2 \]

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\[ |\beta_n| \]

\[ |t_n| \]

\[ Q_n \]
Realization of “Arbitrary” Far Field

Problem Solution – The Workflow

1. Setup geometry, frequency, and material of the design region \((\Omega, ka, \rho)\).
Problem Solution – The Workflow

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---

Problem Solution – The Workflow

1. Setup geometry, frequency, and material of the design region (∋, ka, ρ).
2. Solve MoM and evaluate associated operators R, L, X, and [K] (with AToM\textsuperscript{10}).
3. Perform lossy characteristic mode decomposition \( \mathbf{X}\mathbf{I}_n = \lambda_n (\mathbf{R} + \mathbf{L}) \mathbf{I}_n \).

Realization of “Arbitrary” Far Field Problem Solution – The Workflow

1. Setup geometry, frequency, and material of the design region ($\Omega$, $ka$, $\rho$).
2. Solve MoM and evaluate associated operators $\mathbf{R}$, $\mathbf{L}$, $\mathbf{X}$, and $[\mathbf{K}]$ (with AToM\textsuperscript{10}).
3. Perform lossy characteristic mode decomposition $\mathbf{X} \mathbf{I}_n = \lambda_n (\mathbf{R} + \mathbf{L}) \mathbf{I}_n$.
4. Analyze modes, determine those being used. For example, $\forall n : \eta_{\text{rad}, n}(\mathbf{I}_n) > c$.

Problem Solution – The Workflow

1. Setup geometry, frequency, and material of the design region ($\Omega$, $ka$, $\rho$).
3. Perform lossy characteristic mode decomposition $XI_n = \lambda_n (R + L) I_n$.
4. Analyze modes, determine those being used. For example, $\forall n : \eta_{\text{rad},n}(I_n) > e$.
5. Project (full) quadratic forms onto reduced basis, i.e., $I = \sum_k \beta_k I_k$.

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   - Package FunBo from CTU is used\textsuperscript{11}.

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   - Package FunBO from CTU is used\(^{11}\).
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7. Reuse Lagrange multipliers found for the next iteration (fast convergence).
8. Calculate all associated metrics.
9. Construct Pareto frontier (\(E(F, F_0) vs. \eta_{rad}(I)\) for each current in Pareto).


Problem Solution – The Workflow

1. Setup geometry, frequency, and material of the design region ($\Omega$, $ka$, $\rho$).
2. Solve MoM and evaluate associated operators $\mathbf{R}$, $\mathbf{L}$, $\mathbf{X}$, and $[\mathbf{K}]$ (with AToM\textsuperscript{10}).
3. Perform lossy characteristic mode decomposition $\mathbf{X} \mathbf{I}_n = \lambda_n (\mathbf{R} + \mathbf{L}) \mathbf{I}_n$.
4. Analyze modes, determine those being used. For example, $\forall n: \eta_{\text{rad},n}(\mathbf{I}_n) > e$.
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Example #2: Isotropic Far Field

- The same structure and settings as before.
- The desired far field pattern is isotropic in $\hat{\phi}$ polarization, zero in $\hat{\theta}$ polarization.

Two parallel plates.
Example #2: Cost Functions

\[ E(F, F_0) = \eta_{\text{rad}} |F_0 - F|^2 / (2Z_0) \]

- \( E(F, F_0) \)
- \( \eta_{\text{rad}} \)
- \( |F_0 - F|^2 / (2Z_0) \)
- Duality gap \( g_N \)
Example #2: Pareto Frontier

\begin{align*}
\eta_{\text{rad}}(F_0, F) & \quad 10^{-5} \\
E(F_0, F) & \quad 10^{-4} \\
\eta_{\text{rad}} & \quad 10^{-3} \\
E(F_0, F) & \quad 10^{-2} \\
\eta_{\text{rad}} & \quad 10^{-1} \\
E(F_0, F) & \quad 10^0
\end{align*}
Example #2: Pareto Frontier

\[ E(F_0, F') \]

\[ \eta_{\text{rad}} \]

\[ 0.5, 0.6, 0.7, 0.8, 0.9, 1 \]

\[ 0.2, 0.4, 0.6, 0.8, 0.9, 1 \]

\[ 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^0 \]
Solution A, $\eta_{\text{rad}} = 0.977,$
$E = 0.957.$
Solution A, $\eta_{\text{rad}} = 0.977, E = 0.957$.

Solution B, $\eta_{\text{rad}} = 0.886, E = 0.968$. 
Solution A, $\eta_{rad} = 0.977$, $E = 0.957$.

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Solution C, $\eta_{\text{rad}} = 0.208$, $E = 0.977$.

Solution D, $\eta_{\text{rad}} = 0.002$, $E = 0.983$. 
Example #2: Modal Spectra

\[ \eta_{\text{rad}} = 0.977 \]
\[ E = 0.957 \]

\[ \eta_{\text{rad}} = 0.886 \]
\[ E = 0.968 \]

\[ \eta_{\text{rad}} = 0.208 \]
\[ E = 0.977 \]

\[ \eta_{\text{rad}} = 0.002 \]
\[ E = 0.983 \]
Solution – Problem #2

How to approach Problem #2? (Phase of $F_0$ is arbitrary).

$$\text{minimize}_{I} \| |F_0| - |F(I)| \|^2$$

subject to $\eta_{\text{rad}}(I) \leq x$

- Suddenly, from easy problem we face an unsolvable one…
Solution – Problem #2

How to approach Problem #2? (Phase of $F_0$ is arbitrary).

\[
\begin{align*}
\text{minimize} & \quad \| F_0 \| - \| F(I) \|^2 \\
\text{subject to} & \quad \eta_{\text{rad}}(I) \leq x
\end{align*}
\]

Suddenly, from easy problem we face an unsolvable one...

Some tricks as before, grouping both constraints together, and the phase is taken as an unknown:

\[
\begin{align*}
\text{minimize} & \quad -\frac{1}{Z_0} \text{Re} (I^H [K]^H \Lambda \text{diag} \{ F_0 \} \ p) \\
\text{subject to} & \quad \frac{1}{2} I^H L I = \delta \\
& \quad \frac{1}{2} I^H R I = 1
\end{align*}
\]

where $p = [p_k]$, $p_k = \exp \{ \text{j} \phi_n \}$. 

Miloslav Čapek
The Approach Taken…

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- Hierarchical clusterization of phase diagram via K-means clustering over Lebedev.

1 cluster in $\vartheta$ (8 points skipped).
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The Approach Taken...

- Observation that the bounds have smooth currents is utilized.
- Hierarchical clusterization of phase diagram via K-means clustering over Lebedev.
- Performed for $\vartheta$ and $\varphi$ independently. Zeros of $F_0$ can be skipped.
- \texttt{fmincon} ($\mathbf{p}$) & QCQP ($\mathbf{I}$) co-simulation.
- \texttt{fmincon} can be replaced by, \textit{e.g.}, manifold optimization\textsuperscript{12}.

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\textsuperscript{12}https://www.manopt.org/
Future Outlook – Test Cases

1. What is the cost to replicate far field of one antenna on another (electrically smaller/etc.)?
2. How closely can be, e.g., spherical harmonics radiated by a planar structure?
3. Masked far field (zeros at some places).
4. Close investigation of isotropic radiator (take a spherical shell – no-hair theorem, etc.).
5. What is the cost of pencil beam of different design regions on various parameters?
6. Use projection to port voltages as the only controllable quantities.
7. ...
Concluding Remarks

Far field optimality

- Good problem to think of.
- Mixture of QCQP with other optimization routines.
- Many possible applications...
Concluding Remarks

Far field optimiality
- Good problem to think of.
- Mixture of QCQP with other optimization routines.
- Many possible applications.

Topics of ongoing research
- To treat Problem #2 (with phase) effectively.
- To try many test cases.
- Investigate cost in Q-factor, excitation constraints.
- Apply port-mode representation (for arrays).
Questions?
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The presentation is available at capek.elmag.org

Acknowledgment: To my wife for letting me go :)