Shape Regularization and Acceleration of Topology Optimization via Point Group Symmetries

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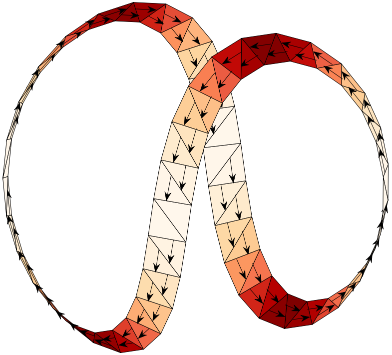
Outline

1. Introduction
2. Symmetry Matrices
3. Examples
4. Concluding Remarks

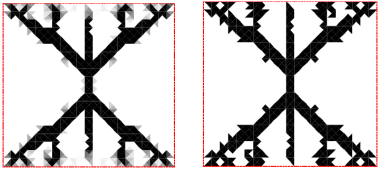


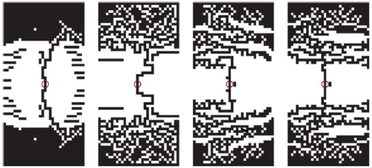
Symmetric magnetic trap in the form of baseball seam (use motif in one of the octants and replicate with $\sigma_{\mathrm{z}} C_{4\mathrm{z}}$ recursively three times).

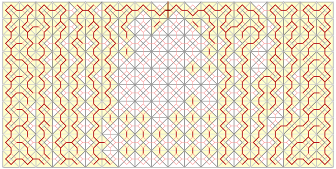
The picture of magnetic trap kindly provided by Jakub Liska from CTU in Prague.

Symmetry – Common Characteristic of EM Designs

(Some pictures taken from internet)







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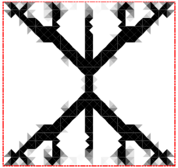
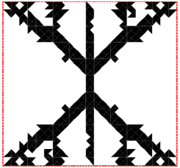
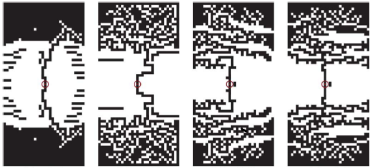
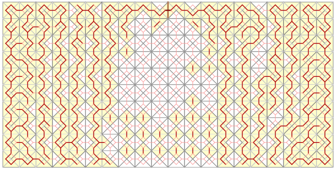




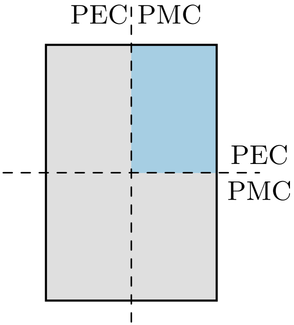


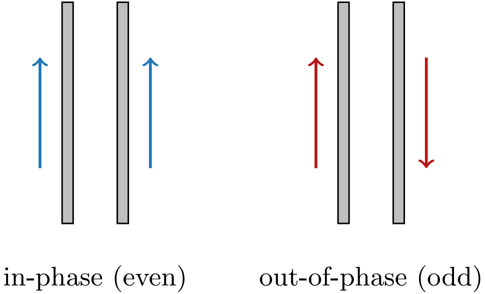


Symmetricity is an important characteristic of many antenna and microwave devices...

Symmetries

- Utilized as PEC/PMC planes, odd/even modes...
- Simplify an EM problem and reduce computational burden.





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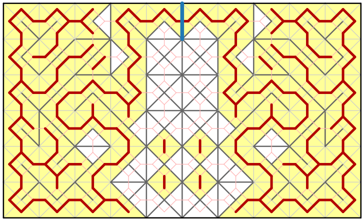
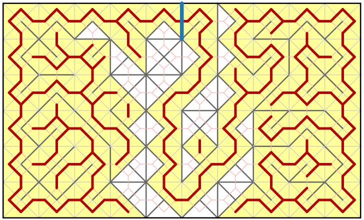
In this work, method of moments paradigm is considered

1/ To enforce symmetries.

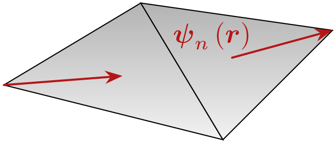
2/ To prefer symmetries.





Application Domain

$$\boldsymbol{J}(\boldsymbol{r}) \approx \sum_n I_n \boldsymbol{\psi}_n(\boldsymbol{r})$$



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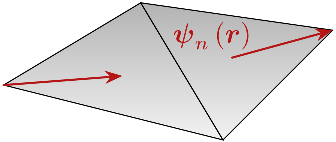
$$\mathbf{Z}\mathbf{I} = \mathbf{V}$$



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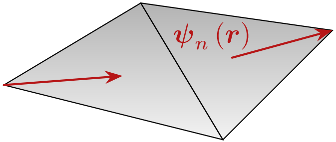
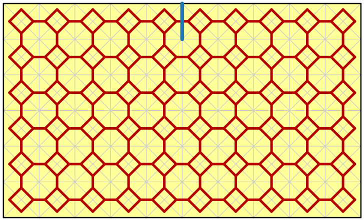
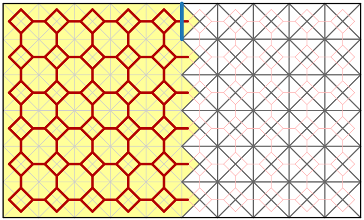


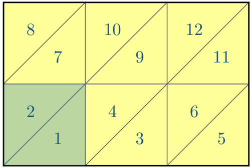




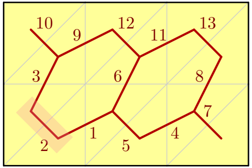
Parameterization of the Problem

- graph-theory analogy

discretization



basis functions



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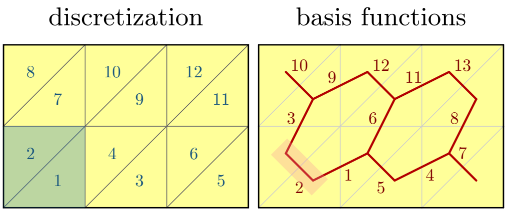

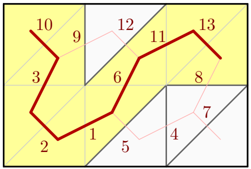

$$\mathbf{t} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$\mathbf{g} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

Mapping Matrices

1. Construct matrices for each symmetry operator R:

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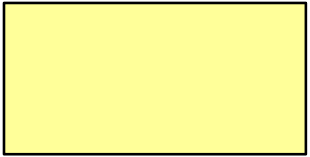
4. Combine symmetries (Note: rigorously, works for 1D irreducible representations only):

$$\mathbf{C}_i = \sum_n \chi_{n,i}\mathbf{C}_n(R_n) \xrightarrow{\text{orth}} \widehat{\mathbf{C}}$$

Example: Mapping Matrices



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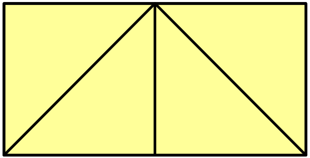

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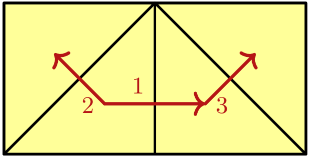

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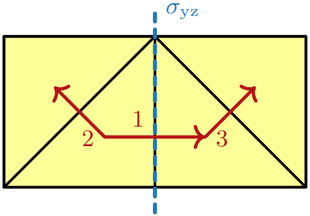

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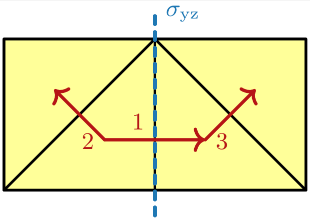

$$\mathbf{C}\left(E\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{C}\left(\sigma_{\mathrm{yz}}\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

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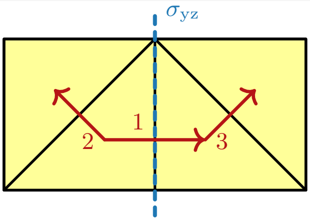

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Irreducible representation of Cs group (odd–even):

	E	σ_{yz}
A'	$+1$	$+1$
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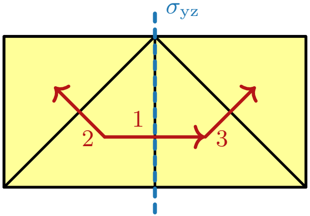

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Let us try the even (PMC) solution (A': $\chi_1 = +1$, $\chi_2 = +1$)

$$\mathbf{C}_{A'} = 1 \times \mathbf{C}\left(E\right) + 1 \times \mathbf{C}\left(\sigma_{\mathrm{yz}}\right) \xrightarrow{\text{orth}} \widehat{\mathbf{C}}_{A'} = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Reduction of Operator Matrices

For any basis functions (both piece-wise and entire-domain):

$$\widehat{\mathbf{A}} = \widehat{\mathbf{C}}^{\mathrm{T}} \mathbf{A} \widehat{\mathbf{C}}$$

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Reduction of the EM problem as:

$$\mathbf{Z}\mathbf{I} = \mathbf{V} \longrightarrow \widehat{\mathbf{Z}}\widehat{\mathbf{I}} = \widehat{\mathbf{V}}, \quad \mathbf{Z} \in \mathbb{C}^{N \times N}, \quad \widehat{\mathbf{Z}} \in \mathbb{C}^{M \times M}, \quad M < N$$

Example 1: Compressed Matrix Decomposition and Inversion

$$\widehat{\mathbf{A}}\widehat{\mathbf{I}}_n = \lambda_n \widehat{\mathbf{I}}_n$$

$$\widehat{\mathbf{I}} = \widehat{\mathbf{Z}}^{-1}\widehat{\mathbf{V}}$$

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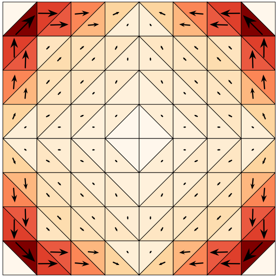
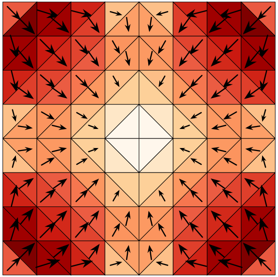
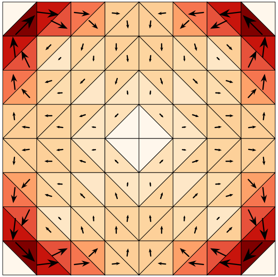
Example: First three characteristic modes following all symmetry operations:

Example 2: Symmetry-Enforced Shape Optimization

$$\begin{aligned} \underset{\widehat{\mathbf{g}}}{\text{minimize}} \quad & f(\widehat{\mathbf{I}}, \widehat{\mathbf{g}}) \\ \text{subject to} \quad & \widehat{\mathbf{Z}}\left(\widehat{\mathbf{g}}\right)\widehat{\mathbf{I}} = \widehat{\mathbf{V}} \end{aligned}$$

- Complexity spans from $\mathcal{O}(M^3)$ to $\mathcal{O}(p^M)$.

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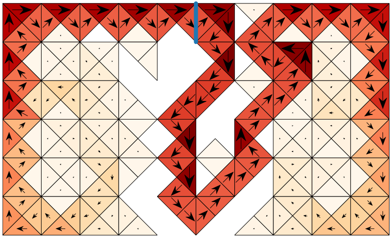
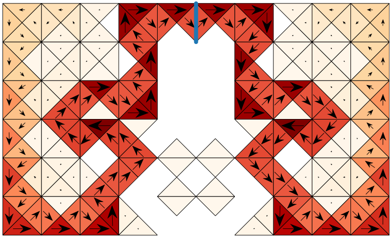


$\widehat{\mathbf{C}} = \mathbf{E}$ (no symmetries)

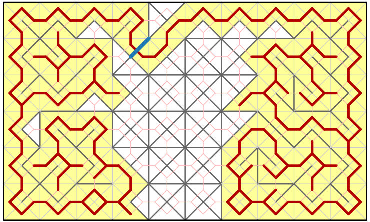


$\mathbf{C}(\sigma_{\mathrm{yz}}) \longrightarrow \widehat{\mathbf{C}}$ (one reflection)

Example 3: Symmetry-Preferred Shape Optimization

$$r_{\mathrm{sym}}(R, \mathbf{g}) = \frac{1}{N} \left\| |\mathbf{C}(R)\mathbf{g}| - \mathbf{g} \right\|_1$$

$$f(\mathbf{g}) = Q\left(\mathbf{I}\left(\mathbf{g}\right)\right) + w\, r_{\mathrm{sym}}\left(\mathbf{g}\right), \quad r_{\mathrm{sym}} \in [0, 1]$$



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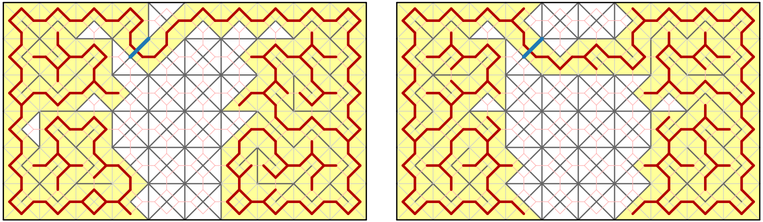

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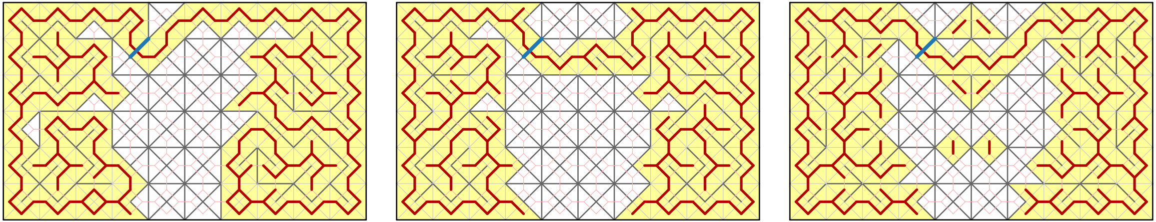

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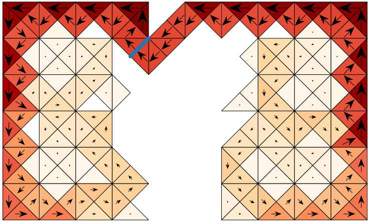

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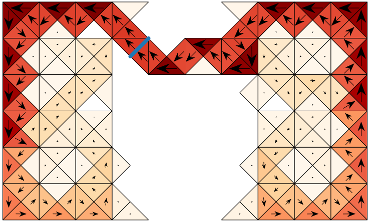

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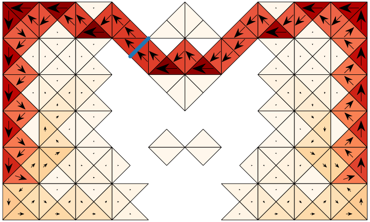
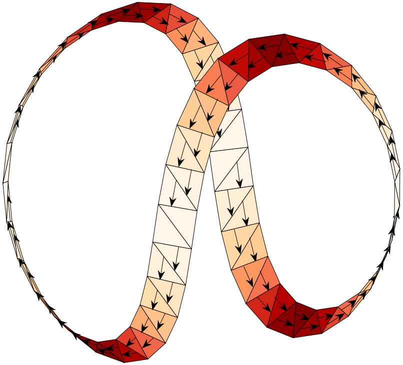


$w = 1$

Concluding Remarks

Implemetation of Symmetries

- Based on point-group theory.
- Versatile and general (2D, 3D, . . .).
- Applicable to problem solution, decomposition, or optimization.
- Easy to add to any method-of-moments code.



Magnetic trap for neutral particles: Symmetries $(\sigma_z C_{4z})^n$ with $n = \{1, 2, 3\}$ applied to reduce the number of unknowns to $1/4$.

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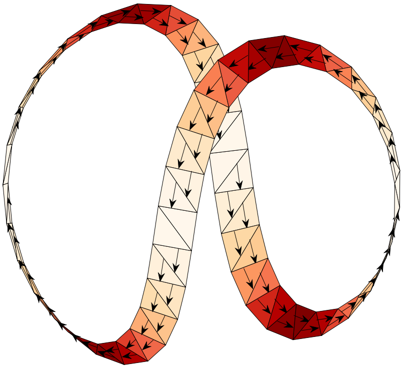
Features

- Enforcing only a given type of solution and regularizing a shape.
- Accelerating a solution by decreasing number of optimization variables.



Magnetic trap for neutral particles: Symmetries $(\sigma_z C_{4z})^n$ with $n = \{1, 2, 3\}$ applied to reduce the number of unknowns to 1/4.

Questions?

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version 1.0
The presentation is available at ▶ capek.elmag.org

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