Shape Regularization and Acceleration of Topology Optimization via Point Group Symmetries

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Symmetric magnetic trap in the form of baseball seam (use motif in one of the octants and replicate with $\sigma_z C_{4z}$ recursively three times).

The picture of magnetic trap kindly provided by Jakub Liska from CTU in Prague.
Symmetry – Common Characteristic of EM Designs
(Some pictures taken from internet)
Symmetricity is an important characteristic of many antenna and microwave devices...
Utilized as PEC/PMC planes, odd/even modes...

Simplify an EM problem and reduce computational burden.

in-phase (even)  out-of-phase (odd)
Symmetries

- Utilized as PEC/PMC planes, odd/even modes...
- Simplify an EM problem and reduce computational burden.

In this work, method of moments paradigm is considered

1/ To enforce symmetries.

2/ To prefer symmetries.
Application Domain

\[ J(r) \approx \sum_n I_n \psi_n(r) \]
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\[ ZI = V \]
$$J(r) \approx \sum_n I_n \psi_n(r)$$

$$ZI = V$$
Parameterization of the Problem

- graph-theory analogy

\[
g = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}^T
\]
\[
t = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{bmatrix}^T
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t = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}^T
\]
1. Construct matrices for each symmetry operator $R$:

$$R\psi_m (r) = \sum_{n=1}^{N} c_{mn}(R) \psi_n (r)$$
Mapping Matrices

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2. Store the entries into matrices $C$:

$$C (R) = [c_{mn} (R)]$$
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3. Delete one of the $(m,n)$-th columns (reduce to linear independent vectors):

$$C \xrightarrow{\text{orth}} \hat{C}$$
Symmetry Matrices

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4. Combine symmetries (Note: rigorously, works for 1D irreducible representations only):

$$ C_i = \sum_{n} \chi_{n,i} C_n (R_n) \xrightarrow{\text{orth}} \hat{C} $$
Example: Mapping Matrices

\[
C(E) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix},
\]

\[
C(\sigma_{yz}) = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

Irreducible representation of Cs group (odd–even):

\[
E \sigma_{yz} \quad A'_{+1} + A''_{+1} - 1
\]

Let us try the even (PMC) solution \((A': \chi_1 = +1, \chi_2 = +1)\)

\[
C(A') = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix} \times C(E) + 1 \times C(\sigma_{yz}) \quad \text{orth} \rightarrow b
\]

\[
C(A') = \sqrt{2} \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}
\]
Example: Mapping Matrices

$$\sigma_{yz}$$

$$C(E) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C(\sigma_{yz}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Irreducible representation of $C_s$ group (odd–even):

$$E \sigma_{yz} A'_{+1} +1 \ A''_{+1} -1$$

Let us try the even (PMC) solution ($A': \chi_1 = +1, \chi_2 = +1$)

$$C A' = 1 \times C(E) + 1 \times C(\sigma_{yz}) \rightarrow b$$

$$C A' = \sqrt{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$
Example: Mapping Matrices

\[
C(σ_{yz}) = \begin{pmatrix}
-1 & 0 & 0 \\
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Irreducible representation of Cs group (odd–even):

\[E σ_{yz} A^\prime +1 +1 A^\prime\prime +1 -1\]

Let us try the even (PMC) solution (\(A^\prime\): \(χ_1 = +1, χ_2 = +1\))

\[C A^\prime = 1 \times C(E) + 1 \times C(σ_{yz})\]

\[\rightarrow b\]

\[C A^\prime = \sqrt{2} \begin{pmatrix} 0 \\
1 \\
1 \end{pmatrix}\]
Example: Mapping Matrices

\[ C(E) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]
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Irreducible representation of Cs group (odd–even):

\[ E \sigma_{yz} A' + 1 + 1 A'' + 1 - 1 \]

Let us try the even (PMC) solution (\( A': \chi_1 = +1, \chi_2 = +1 \))

\[ C A' = 1 \times C(E) + 1 \times C(\sigma_{yz}) \rightarrow b \]

\[ C A' = \sqrt{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \]
**Example: Mapping Matrices**

\[
\mathbf{C}(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{C}(\sigma_{yz}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
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Irreducible representation of Cs group (odd–even):

<table>
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<tr>
<th></th>
<th>( E )</th>
<th>( \sigma_{yz} )</th>
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</thead>
<tbody>
<tr>
<td>( A' )</td>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>( A'' )</td>
<td>+1</td>
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Let us try the even (PMC) solution ($A'$: $\chi_1 = +1$, $\chi_2 = +1$)

\[
C_{A'} = 1 \times C(E) + 1 \times C(\sigma_{yz}) \xrightarrow{\text{orth}} \hat{C}_{A'} = \frac{\sqrt{2}}{2} \begin{pmatrix}
0 \\
1 \\
1 \\
\end{pmatrix}
\]
Reduction of Operator Matrices

For any basis functions (both piece-wise and entire-domain):

\[ \hat{A} = \hat{C}^T A \hat{C} \]

\[ \hat{b} = \hat{C}^T b \]

\[ \hat{g} = \hat{C}^T g \]
Reduction of Operator Matrices

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Reduction of the EM problem as:

\[ Z \mathbf{I} = V \rightarrow \hat{Z} \hat{I} = \hat{V}, \quad Z \in \mathbb{C}^{N \times N}, \quad \hat{Z} \in \mathbb{C}^{M \times M}, \quad M < N \]
Example 1: Compressed Matrix Decomposition and Inversion

\[ \hat{A}\hat{I}_n = \lambda_n \hat{I}_n \]

\[ \hat{I} = \hat{Z}^{-1}\hat{V} \]
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**Example:** First three characteristic modes following all symmetry operations:
Example 2: Symmetry-Enforced Shape Optimization

\[
\begin{align*}
\text{minimize} & \quad f(\hat{\mathbf{I}}, \hat{\mathbf{g}}) \\
\text{subject to} & \quad \hat{\mathbf{Z}}(\hat{\mathbf{g}}) \hat{\mathbf{I}} = \hat{\mathbf{V}}
\end{align*}
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- Complexity spans from $O(M^3)$ to $O(p^M)$. 

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- Complexity spans from $O(M^3)$ to $O(p^M)$.

\[\hat{C} = E \text{ (no symmetries)} \quad \quad \text{C}(\sigma_{yz}) \rightarrow \hat{C} \text{ (one reflection)}\]
Example 3: Symmetry-Preferred Shape Optimization

$$r_{\text{sym}}(R, g) = \frac{1}{N} \left\| |C(R)g| - g \right\|_1$$

$$f(g) = Q(I(g)) + w r_{\text{sym}}(g), \quad r_{\text{sym}} \in [0, 1]$$

$$w = 0$$
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\[ w = 0.3 \]
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Concluding Remarks

Implementation of Symmetries

- Based on point-group theory.
- Versatile and general (2D, 3D, ...).
- Applicable to problem solution, decomposition, or optimization.
- Easy to add to any method-of-moments code.

Magnetic trap for neutral particles: Symmetries 
\((\sigma_z C_{4z})^n\) with \(n = \{1, 2, 3\}\) applied to reduce the number of unknowns to 1/4.
### Concluding Remarks

#### Implementation of Symmetries

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#### Features

- Enforcing only a given type of solution and regularizing a shape.
- Accelerating a solution by decreasing number of optimization variables.

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Magnetic trap for neutral particles: Symmetries $$(\sigma_z C_4)^n$$ with $n = \{1, 2, 3\}$$ applied to reduce the number of unknowns to $1/4$. 

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Questions?

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The presentation is available at capek.elmag.org

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