Modal Crossing Treatment Using Group Theory

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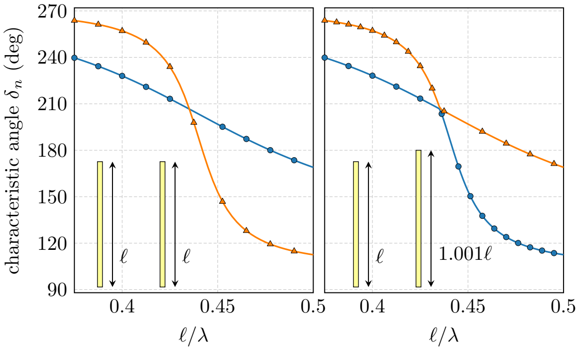
Outline

1. Motivation
2. Point Group Application
3. Examples
4. Conclusion



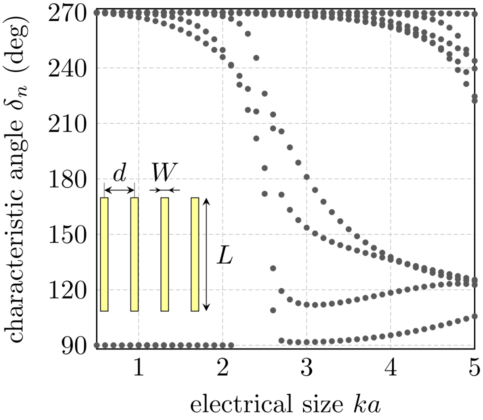
The first odd and even characteristic modes in the vicinity of their resonance. Two dipoles of the same length (left) and with slightly different lengths (right).

Motivation

- Generalized eigenvalue problem[1]:

$$\mathbf{A}(p)\,\mathbf{I}_n(p) = \lambda_n(p)\,\mathbf{B}(p)\,\mathbf{I}_n(p)\,.$$

$$\delta_n = 180^\circ\left(1 - \frac{1}{\pi}\arctan(\lambda_n)\right)$$



Raw modes of four dipoles.

[1] Illustrated on characteristic modes: $\mathbf{A} = \mathbf{X}$, $\mathbf{B} = \mathbf{R}$, $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$ is an impedance matrix.

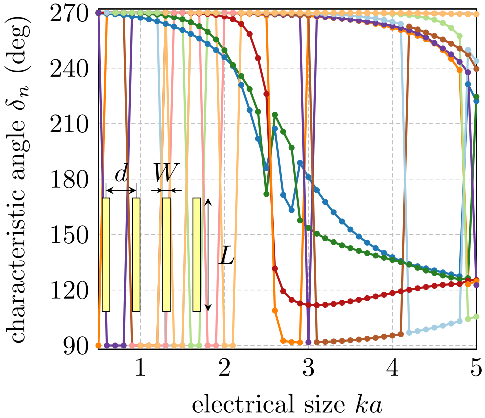
Motivation

- Generalized eigenvalue problem[1]:

$$\mathbf{A}(p)\,\mathbf{I}_n(p) = \lambda_n(p)\,\mathbf{B}(p)\,\mathbf{I}_n(p)\,.$$

- Tracking is required to get $\lambda_n(p)$ as smooth function over p.

$$\delta_n = 180^\circ \left(1 - \frac{1}{\pi}\arctan(\lambda_n)\right)$$



Untracked modes of four dipoles.

[1]Illustrated on characteristic modes: $\mathbf{A} = \mathbf{X}$, $\mathbf{B} = \mathbf{R}$, $\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X}$ is an impedance matrix.

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$$\mathbf{A}(p)\,\mathbf{I}_n(p) = \lambda_n(p)\,\mathbf{B}(p)\,\mathbf{I}_n(p)\,.$$

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- Current techniques are based on correlation methods.

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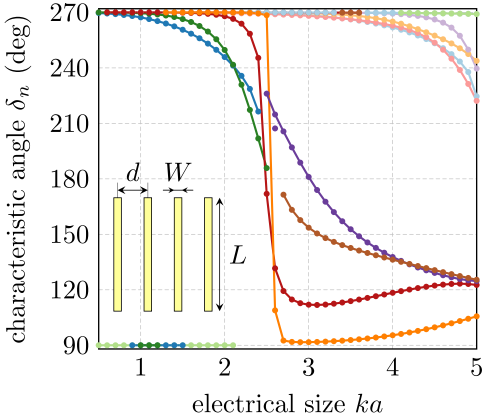

Tracked modes of four dipoles.

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Motivation

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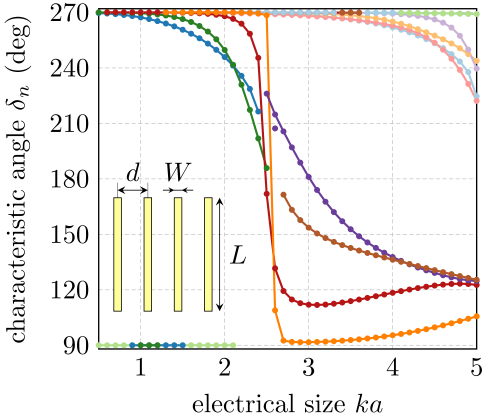
- Tracking is required to get $\lambda_n(p)$ as smooth function over p.
- Current techniques are based on correlation methods.
- Are the results tracked properly?

$$\delta_n = 180^\circ\left(1 - \frac{1}{\pi}\arctan(\lambda_n)\right)$$



Tracked modes of four dipoles.

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Initial Theorems

von Neumann-Wigner theorem[2]

Crossing of modes' eigenvalue traces are possible only between modes belonging to the different (irreducible) representation. Eigenvalue traces of modes in the same irreducible representation can not cross.

[2] J. von Neumann and E. Wigner, "On the behaviour of eivenvalues in adiabatic processes," in *Quantum Chemistry: Classic Scienti c Papers*. Singapore: World Scientific, 2000

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- Having non-symmetric structure[3]: "Characteristic values $\lambda_n(p)$ of non-symmetric structures can not cross each other."

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- Having non-symmetric structure[3]: "Characteristic values $\lambda_n(p)$ of non-symmetric structures can not cross each other."
- Question: How to divide modes of symmetrical structures into irreducible representations?

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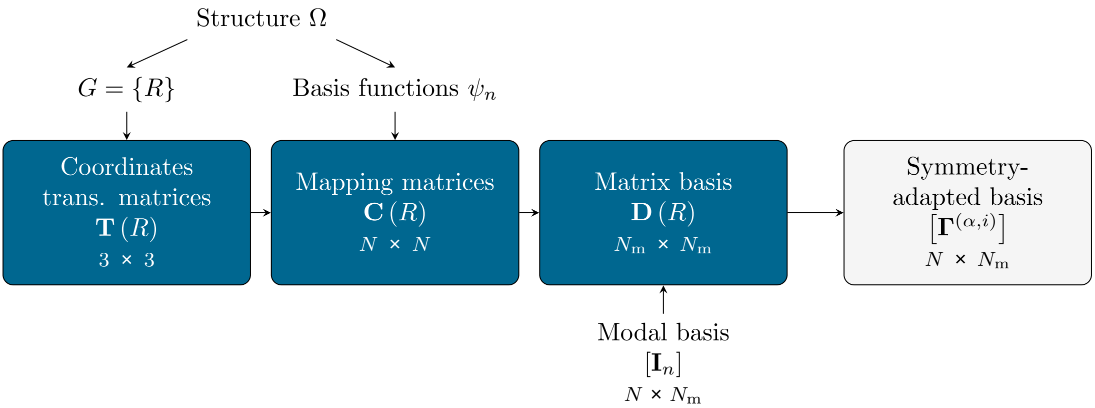
Classification of Modes[4]



[4] M. Masek, M. Capek, L. Jelinek, *et al.*, "Modal tracking based on group theory," , Dec. 7, 2018, eprint arXiv: 1812.03006. [Online]. Available: https://arxiv.org/abs/1812.03006

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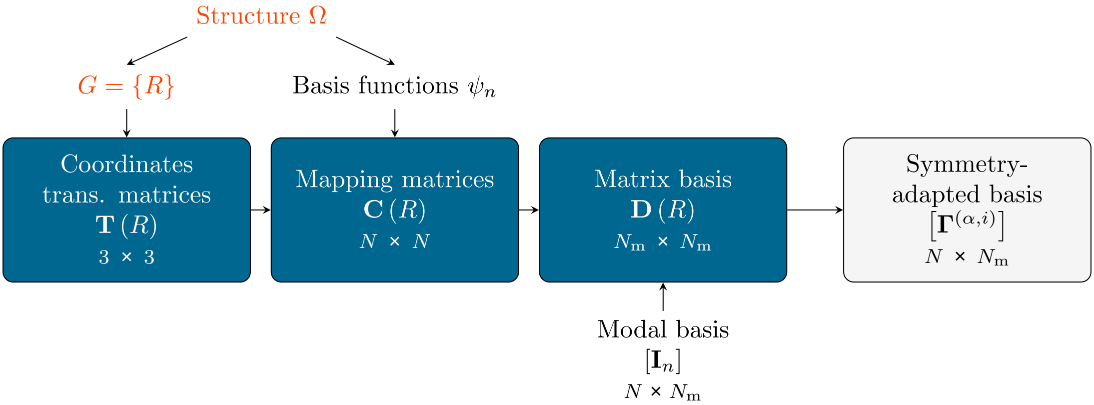

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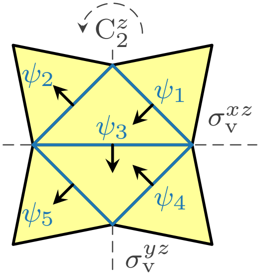
Classification of Modes[4]

- Symmetrical structure with symmetrical mesh.
- Let us suppose known set of symmetry operations $G = \{R\}$.
  - E – identity operation,
  - σ_{v}^{yz} – reflection by yz plane,
  - σ_{v}^{xz} – reflection by xz plane,
  - C_2^z – rotation by π around z axes.



Structure with five basis functions (Group $\mathrm{C}_{2\mathrm{v}}$).

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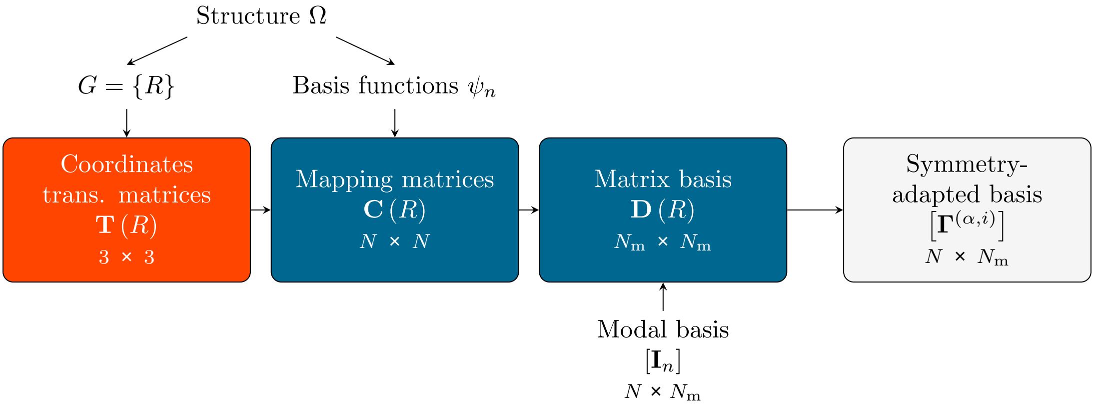

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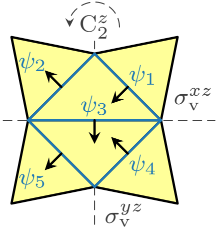
Classification of Modes[4]

- Each operation R is represented by 3×3 coordinate transform matrix $\mathbf{T}(R)$.



$$\mathbf{T}(\mathrm{E}) = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

$$\mathbf{T}(\sigma_{\mathrm{v}}^{yz}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

$$\mathbf{T}(\sigma_{\mathrm{v}}^{xz}) = \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

$$\mathbf{T}(\mathrm{C}_2^z) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

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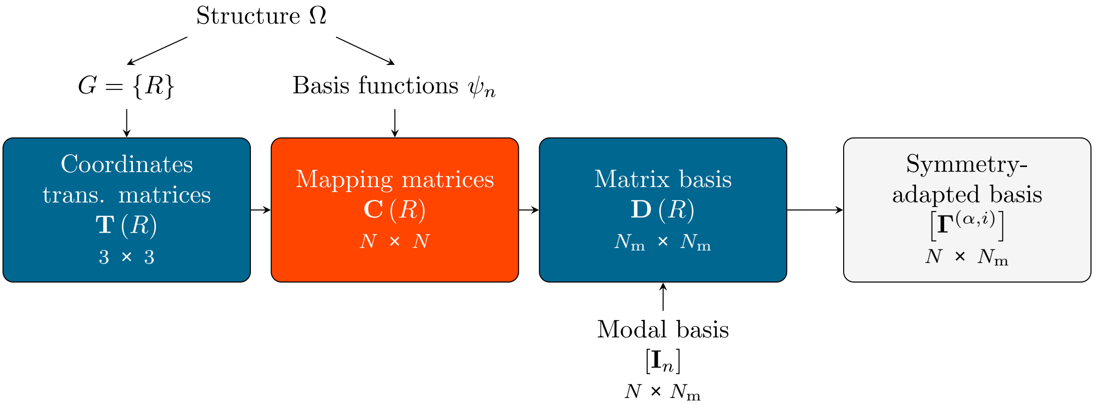

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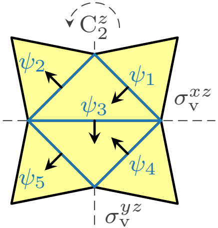
Classification of Modes[4]

- Construction[5] of mapping matrices $\mathbf{C}\left(R\right)$.



$$\mathbf{C}\left(\sigma_{\mathrm{v}}^{yz}\right) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{C}\left(\sigma_{\mathrm{v}}^{xz}\right) = \begin{bmatrix} 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 1 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}\left(\mathrm{C}_2^z\right) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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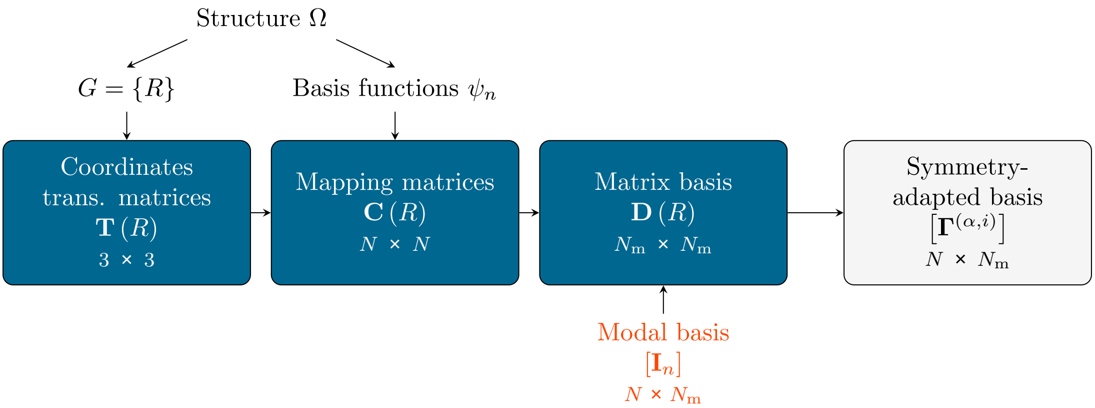
[5] An identity operation E ins not shown explicitly.

Classification of Modes[4]



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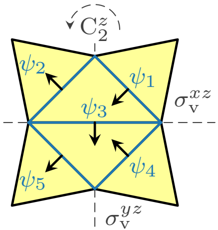
Classification of Modes[4]

- Compute modal basis $[\mathbf{I}_n]$.

$$\mathbf{X}\mathbf{I}_n = \lambda_n \mathbf{R}\mathbf{I}_n$$

$$[\mathbf{I}_n] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1.76 & 0 & 0 & 0 & 1.62 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



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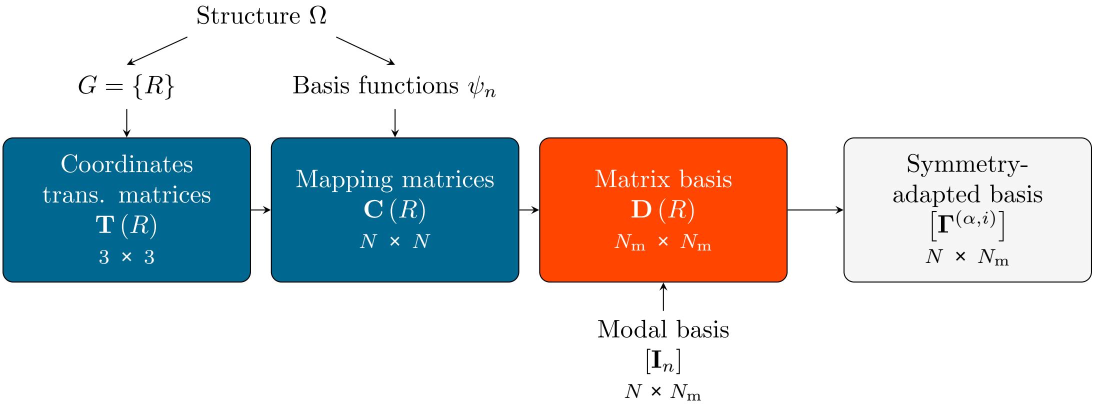

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Classification of Modes[4]

- Representation in modal basis:

$$\mathbf{C}(R)[\mathbf{I}] = [\mathbf{I}]\,\mathbf{D}(R)$$

$$[\mathbf{I}_n] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1.76 & 0 & 0 & 0 & 1.62 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{D}\left(\sigma_{\mathrm{v}}^{yz}\right) = \begin{bmatrix} +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{bmatrix}$$

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$$\mathbf{D}\left(\mathrm{C}_2^{z}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Classification of Modes[4]

- Representation in modal basis:

$$\mathbf{C}(R)[\mathbf{I}] = [\mathbf{I}]\mathbf{D}(R)$$

- Classification of modes – unique combinations of blocks in $\mathbf{D}(R)$ matrices.

$$[\mathbf{I}_n] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1.76 & 0 & 0 & 0 & 1.62 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{D}(\sigma_{\mathrm{v}}^{yz}) = \begin{bmatrix} +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{bmatrix}$$

$$\mathbf{D}(\sigma_{\mathrm{v}}^{xz}) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

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Classification of Modes

- Character table[6]
  - Values are traces of blocks.

irrep α	E	σ_v^{yz}	σ_v^{xz}	C_2^z
A_1	+1	+1	+1	+1
A_2	+1	−1	−1	+1
B_1	+1	−1	+1	−1
B_2	+1	+1	−1	−1

Character table for C_{2v} group.

$$\mathbf{D}\left(\sigma_v^{yz}\right) = \begin{bmatrix} +1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{bmatrix}$$

$$\mathbf{D}\left(\sigma_v^{xz}\right) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

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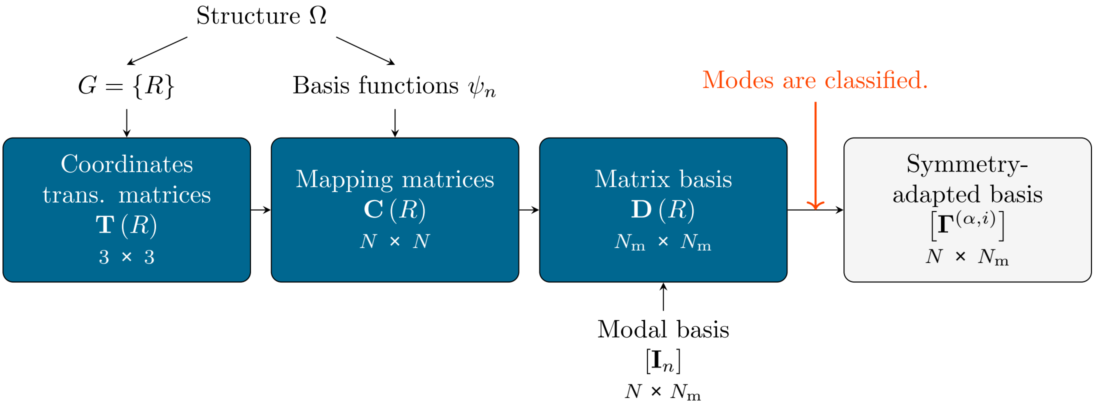
[6] R. McWeeny, *Symmetry: An Introduction to Group Theory and Its Applications*. London: Pergamon Press, 1963

Classification of Modes[4]



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Modal Tracking

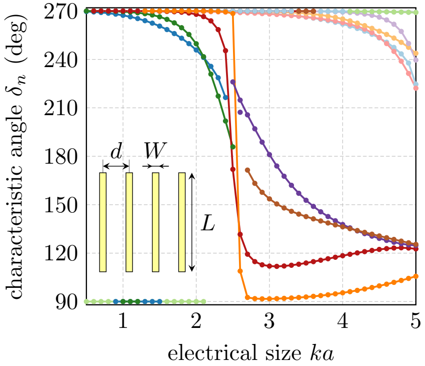
Tracking simplification

Modes within each irreducible representation can not cross.

$\downarrow$

Tracking algorithm can be replaced by sort.

Four-dipole Array

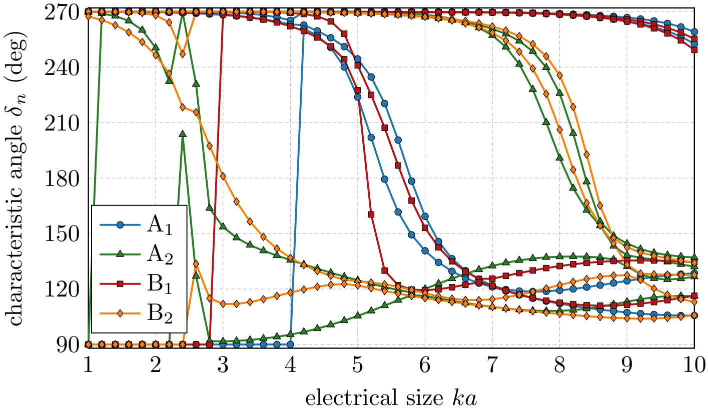


Tracked modes of four dipoles.

group C_{2v}	ka_{res}	$\mathbf{I}_1$	$\mathbf{I}_2$
A_2	2.8	↓↓↑↑	↑↓↑↓
B_2	2.8	↑↑↑↑	↑↓↓↑
A_1	5.6	↑↑↑↑ ↓↓↓↓	↓↑↑↓ ↑↓↓↑
B_1	5.6	↑↑↓↓ ↓↓↑↑	↑↓↑↓ ↓↑↓↑

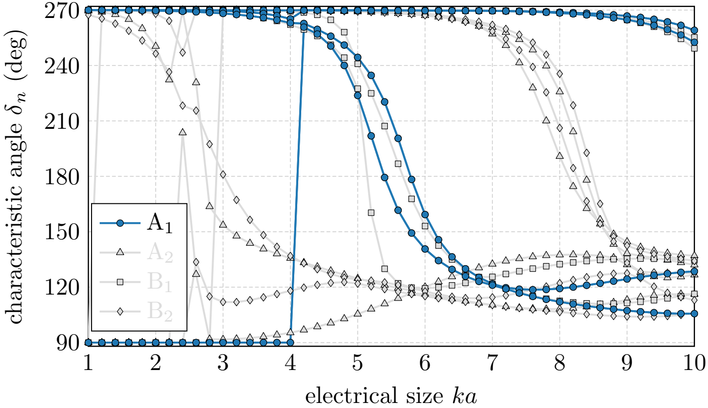
Current distribution of the first two modes of each irrep of the four-dipole array. Only directions of currents on each dipole is depicted, different amplitudes are not considered.

Four-dipole Array



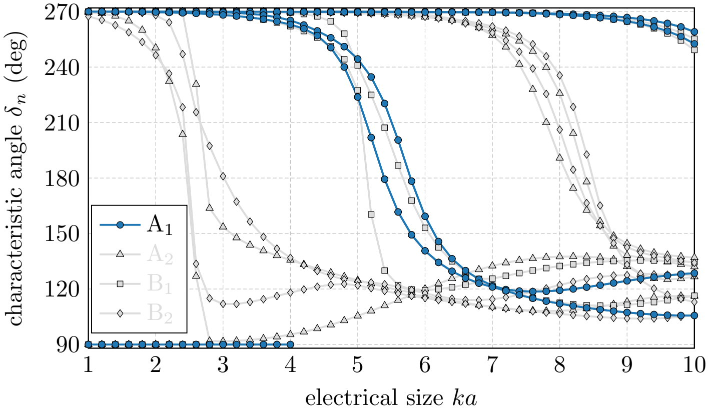
Tracked characteristic modes of four-dipole array.

Four-dipole Array



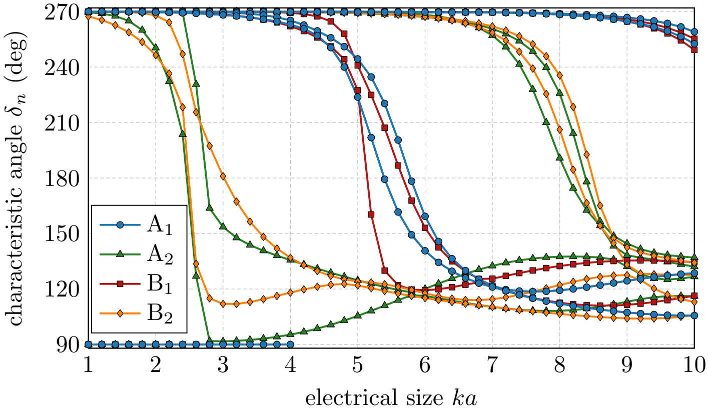
Tracked characteristic modes of four-dipole array.

Four-dipole Array



Tracked characteristic modes of four-dipole array.

Four-dipole Array



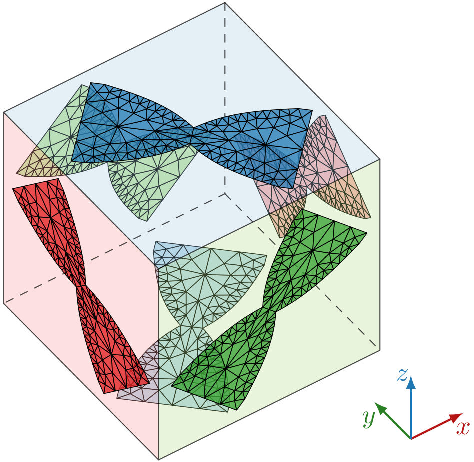
Tracked characteristic modes of four-dipole array.

A Cubic Array

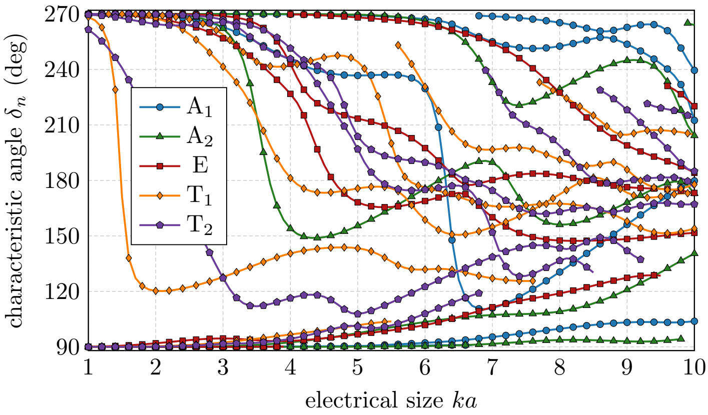
- Six symmetrical bowtie antennas.
- Point group T_d.

T_d	E	$8C_3$	$3C_2$	$6S_4$	$6\sigma_d$
A_1	+1	+1	+1	+1	+1
A_2	+1	+1	+1	−1	−1
E	+2	−1	+2	0	0
T_1	+3	0	−1	+1	−1
T_2	+3	0	−1	−1	+1

Character table for point group T_d.



A Cubic Array



Tracked characteristic modes of bowtie cube.

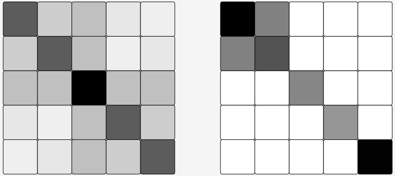
Conclusion

- Deterministic and exact solution of modal tracking via the theory of point groups.
- Other applications:
  - Symmetry-adapted basis to block-diagonalize impedance matrix[4].
  - Uncorrelated channels for MIMO systems.

Block-diagonalization of the impedance matrix

- Symmetry-adapted basis $\mathbf{\Gamma}^{(\alpha,i)}$

$$\mathbf{Z}^{\mathrm{d}} = \left[\mathbf{\Gamma}^{(\alpha,i)}\right]^{\mathrm{T}} \mathbf{Z} \left[\mathbf{\Gamma}^{(\alpha,i)}\right]$$



- Each block can be computed separately:
  - Modes solely from species (α, i) are computed.
  - Reduction of computational time.

[4] M. Masek, M. Capek, L. Jelinek, *et al.*, "Modal tracking based on group theory," , Dec. 7, 2018, eprint arXiv: 1812.03006. [Online]. Available: https://arxiv.org/abs/1812.03006

Questions?

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April 4, 2019
EuCAP 2019, Krakow, Poland
version 1.0
The presentation is downloadable at
michal-masek.cz

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