3D System of Randomly Detuned Ring Resonators

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Abstract— A homogenization procedure aimed at analyzing a metamaterial designed as a system composed of point magnetic dipoles is presented in this paper. These dipoles represent broad side coupled split ring resonators and/or conducting loops terminated by capacitors. A non-precision fabrication process causes dispersion of the geometrical and material parameters of the resonators, and thus dispersion of their resonance frequencies. The homogenization procedure takes into account random distribution of resonator resonance frequencies with the normal probability distribution. The resonant band of the effective permeability of the metamaterial is significantly widened, the values of both real and imaginary parts of the permeability are reduced. The band of negative permeability real part disappears at high dispersion. The paper also comments on the design of a lens aimed at improving the imaging properties of the MRI system. This lens is composed of cells with cubic symmetry. The proposed homogenization method is verified by measuring the scattering parameters of the metamaterial in a rectangular waveguide, and by proving the isotropic response of the metamaterial of the lens.

Index Terms—Homogenization; metamaterial; magnetic dipole; ring resonator; effective permeability.

I. INTRODUCTION

Metamaterials used in the microwave frequency range are designed as 3D periodic systems of planar resonant elements showing negative electric and/or magnetic polarizabilities. The material and geometric parameters of the resonators determine their resonance frequency, so variations of these parameters can be treated by accounting resonators with dispersed resonance frequencies [1]. Resonance frequency dispersion results in widening the metamaterial response and reducing the values of its effective parameters. The band of negative effective permittivity and/or permeability can even disappear.

Attention has been paid on the literature to studies of the behavior of composite dielectric or metal-dielectric materials showing some degree of randomness [2,3]. However, less attention has been paid to investigations of metamaterials with randomly positioned resonant elements or with randomly dispersed material or geometric parameters. A metamaterial composed basically of a periodic system of cut-wire pairs was investigated in [4]. Systems of randomly oriented chiral elements were studied in [5]. A fully experimental investigation of randomness in metamaterials was performed in [6-9]. Another group of works has investigated periodic structures composed of elements differing in their parameters, e.g., in their geometrical dimensions. Gollub et al. [8] used the Bruggeman mixing model and showed widening and finally the disappearance of the band of negative permeability by modifying more and more split ring resonator (SRR) dimensions in a periodic system. Fujii [9] carried out an experimental study on the focusing property of a 2D LH superlens, introducing fluctuation in the capacitance values, gradually increasing the number of modified capacitors, and examining the distortion in the focused electromagnetic images. Papers [1, 10] investigated a 3D periodic system of SRRs. The fluctuations of the SRR parameters were represented by the fluctuations of their resonance frequencies. The authors showed that microscopic averaging can be performed prior to statistical averaging only under the assumption that the resonance frequency is a slowly varying random function of the coordinates. Paper [11] investigated the periodic system of BC-SRRs aligned in one direction, assuming that all the resonators are excited by the same magnetic field.

This work, based on a preliminary study [11], studies the behavior of two 3D periodic systems of ring resonators. The first system was composed of broad side coupled SRRs (BC-SRR) aligned in one direction, and the second system was composed of loop resonators terminated by capacitors and located in cubic cells. These periodic systems are homogenized using the way described by Tretyakov [12], assuming that magnetic dipoles have randomly distributed resonance frequencies. The results of the analysis are verified by measuring transmission of the system of BC-SRRs inserted in the waveguide. Finally, the paper comments on the design procedure for a lens used in the MRI system [13] assuming that the resonators do not have perfectly equal parameters. The homogenization procedure is here modified to take into account a metamaterial with cells of cubic symmetry, as this material must show an isotropic response and this isotropy is proved.

II. HOMOGENIZATION PROCEDURE

The homogenization procedure was first applied to a system composed of BC-SRRs located in a 3D periodic lattice with period a, all aligned in the z direction. Period a is assumed to be much smaller than the free space wavelength. The resonators are approximated by point magnetic dipoles with polarizability [14]

\[ \alpha(\omega, \omega_0) = \frac{A\omega_0^2}{\omega_0^2 - \omega^2 + j\omega\delta}, \]

where \( \omega \) and \( \omega_0 \) are the angular and resonance angular frequencies, respectively, \( A \) is positive amplitude, and \( \delta \) represents losses. The excitation and also the total averaged
The local magnetic field excited by the reference magnetic dipole located at position \(i\). This field can be expressed by [15] p. 413, eq. (9.35). In (2) the fields are summed over all magnetic dipoles, with the exception of \(i = 0\), as the dipole cannot excite itself. The average value of the field excited by the reference magnetic dipole \(\mathbf{H}_0\) is given by [15] p. 188, eq. (5.62).

Local magnetic fields \(\mathbf{H}_i\) determine the magnetic moments of magnetic dipoles located generally in directions defined by unit vectors \(\mathbf{m}_i\) (for dipoles aligned in the \(z\) direction \(\mathbf{m}_i = \mathbf{z}_i\))

\[
\mathbf{m}_i = \alpha_i(\omega_i, \omega_o)(\mathbf{H}_i \cdot \mathbf{m}_i)\mathbf{m}_i.
\]

Now eq. (2) represents the system of algebraic equations for unknown components of vectors \(\mathbf{H}_i\). The analysis presented in [11] was simplified by assuming that all magnetic dipoles have equal moments, i.e., all fields \(\mathbf{H}_i\) are equal. This is approximately valid in the specific case of dipoles aligned in one direction, but not for the metamaterial of the MRI lens analyzed next in this paper.

The effective permeability is determined from the relation

\[
\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0\left(\mathbf{H} + \frac{\mathbf{m}_0}{a}\right)
\]

inserting for the magnetic moment of the reference magnetic dipole (3) one gets

\[
\mu_z(\omega_i, \omega_o) = 1 + \frac{\alpha_i(\omega_i, \omega_o)(\mathbf{H}_0 \cdot \mathbf{m}_0)\mathbf{m}_0z}{a},
\]

where \(\mathbf{m}_0z\) is the \(z\) component of unit vector \(\mathbf{m}_0\). Finally, the effective permeability is determined by integrating (5) multiplied by the distribution function of the resonance frequencies \(\mathcal{F}(\omega_o)\), that is the normal probability distribution.

In the summation of (2), the resonance frequencies are randomly chosen around the central value, independently for each magnetic dipole, and (2) is averaged over a chosen number of realizations. This averaging replaces \(N\) dimensional integration. The resulting process has fast convergence. The summation in (2) is taken over the cube centered at the origin of the coordinate system, with its edge equal to \(4a\). The final sums are averaged over 300 realizations. The cube edge length and the number of realizations have been set large enough to obtain convergent results.

### III. BC-SRRs with Randomly Dispersed Resonance Frequencies

The analyzed periodic system is composed of BC-SRR, investigated in [16], see Fig. 1. The resonators were fabricated on a Rogers RT Duroid 5880 substrate 0.127 mm in thickness with permittivity 2.2 and 0.017 mm copper cladding. The dimensions according to Fig. 1a are: \(R = 1.8\) mm, \(w = 0.7\) mm, \(g = 0.3\) mm. The polarizability of these planar resonators can be described at the low frequency limit by analytical expression (1). Application of the procedure for determining the free space polarizability presented in [17] gives the particular quantities: \(\alpha_k = 1.922 \times 10^{10}\) s\(^{-1}\), \(A = 5 \times 10^8\) m\(^3\), \(\delta = 5.5 \times 10^{-8}\) s\(^{-1}\). The resonance frequencies of 149 BC-SRRs were measured in order to determine their distribution function. This function was fitted by a Gaussian distribution with mean value \(\omega_o = 3.06\) GHz, and with standard deviation \(\sigma = 0.0252\) GHz.

The behavior of a metamaterial composed of the above defined BC-SRRs with dispersed resonance frequencies is documented in Fig. 2. The dispersion curves for the metamaterial of period \(a = 11\) mm are plotted here in dependence on the standard deviation of the resonance frequencies, shown in GHz. An increase in the standard deviation reduces the absolute value of the effective permeability and widens the frequency band around the resonance. For standard deviation greater than \(0.035\) GHz, the band of negative effective permeability disappears. The minimum of \(\mu_{\text{eff}}\) is -0.137 at \(3.145\) GHz with \(\mu_{\text{eff}} = -0.756\) for standard deviation equal to \(0.035\) GHz. The negative maximum of the imaginary part of the effective permeability representing the losses increases with decreasing standard deviation.

The scattering parameters of the metamaterial prism of dimensions 70x70x33 mm assembled by inserting BC-SRRs, Fig. 1b, into three slices of polystyrene, Fig. 1c, were measured in rectangular waveguide R32 with metallic walls in order to verify the theory presented here. The results of these
measurements, see Fig. 3, were compared with the scattering parameters calculated by the CST Microwave Studio, applying the complex permeability determined by the homogenization process presented above. The curve denoted in Fig. 3 CST S21 represents the results of an analysis of a metamaterial composed of BC-SRRs with the distribution of their resonance frequencies determined by standard deviation equal to 0.025 GHz. For completeness, Fig. 3 shows the plot of the real part of the effective permeability, not shown in Fig. 2.

\[ \hat{H}(\theta) = \hat{H}_x \hat{e}_x + \hat{H}_z \hat{e}_z \]

is constant for all angles. The same is valid for the imaginary parts, and for the dependence on both angles \( \theta \) and \( \alpha \) in the field vector and document the isotropic response. The parameters for the resonators defined in the following text.

\[ \mu(\omega) = \sqrt{(\mu'_x(\omega,\theta))^2 + (\mu'_z(\omega,\theta))^2} \]

Figure 3. Measured and calculated scattering parameters of a metamaterial prism with period \( a = 11 \) mm.

It should be pointed out that the results of the presented homogenization method cannot be compared precisely with the measurements for small values of unit cell dimension \( a \), as its applicability assumes that the resonant element is much smaller than period \( a \), and the outer diameter of the resonators used in the experiment is 5 mm.

IV. MRI LENS

The final example of the analysis presents the behavior of a metamaterial made of resonant loops terminated by capacitors for application as a lens for improving the resolution of the 1.5 T MRI system working at 63.87 MHz [13], see Fig. 4a. The lens is composed as a 3D system of loops terminated by capacitors placed into cells of cubic shape to get the isotropic response of the metamaterial [18], see the detail in Fig. 4a. This means that each cell is composed of three loops, as shown in Fig. 4b. The homogenization procedure described in Paragraph 2 was modified accordingly.

\[ \theta = \frac{a}{2} \]

Figure 4. The sketch of a MRI lens [18] (a), the unit cell used in the homogenization process (b).

The homogenization procedure presented here was verified by proving the isotropic response of the metamaterial for the MRI lens [13]. This was done by setting a general direction of field \( \hat{H} \) in (2). Fig. 5 documents the isotropic response in plane \( x-z \), where real part permeabilities \( \mu'_x \) and \( \mu'_z \) are plotted as functions of angle \( \theta \) measured between \( \hat{H} \) and the \( z \) axis. Note that \( \mu'_x = 0 \) at \( \theta = 0 \) deg and conversely \( \mu'_z = 0 \) at \( \theta = 90 \) deg. The “total” permeability calculated as

\[ \mu(\omega) = \sqrt{(\mu'_x(\omega,\theta))^2 + (\mu'_z(\omega,\theta))^2} \]

is constant for all angles. The same is valid for the imaginary parts, and for the dependence on both angles \( \theta \) and \( \alpha \) in the field vector and document the isotropic response. The parameters for the resonators defined in the following text.

\[ \mu(\omega) = \sqrt{(\mu'_x(\omega,\theta))^2 + (\mu'_z(\omega,\theta))^2} \]

Figure 5. The dependence of the real parts of the effective permeabilities in x direction (a), and in z direction (b), on angle \( \theta \) (marked as “th”). Calculated for resonators with resonance frequency equal to 0.0554 GHz.

The resonance frequency of particular resonant loops is 0.0554 GHz. The polarizability of the resonant element that was used is defined by particular parameters in (1):

\[ \omega_0 = 3.48 \times 10^5 \text{s}^{-1}, \quad A = 7.1 \times 10^{-7} \text{m}^3, \quad \delta = 1 \times 10^{-7} \text{s}^{-1} \]

The period of the 3D system equals 15 mm. The total averaged magnetic field has the form \( \mathbf{H} = H_x \hat{e}_x \). Assuming resonators without dispersion of resonance frequencies, the metamaterial is designed to show the real part of the effective permeability equal to -1 at 63.87 MHz, with the relatively low imaginary part \( \mu''_{\text{eff}} = -0.5 \), i.e., with the relatively low losses. The negative maximum of \( \mu''_{\text{eff}} \) is equal to -3.4 at 0.0606 GHz, see Fig. 6. The sensitivity of the real part of the effective

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is constant for all angles. The same is valid for the imaginary parts, and for the dependence on both angles \( \theta \) and \( \alpha \) in the field vector and document the isotropic response. The parameters for the resonators defined in the following text.
permeability on the parameters of the structure is rather small at the working frequency, in comparison with the sensitivity near the negative maximum. Fig. 6 shows that the lens can be designed using resonant elements with the resonance frequency distribution defined by resonance frequency 0.0554 GHz and by standard deviation raised up to 1.5 MHz. The real part of the effective permeability still reaches a value of -1 around frequency 0.0635 GHz, but the imaginary part value starts to grow.

![Figure 6](image_url)  
**Figure 6.** Calculated real part of the effective permeability of the metamaterial for an MRI lens with period a = 15 mm.

V. CONCLUSIONS

Metamaterials are composed as periodic systems of resonant elements. The fabrication process of the elements always has some dispersion of the dimensions and the material parameters that is represented by the corresponding dispersion of the resonance frequencies. The randomness is incorporated into the metamaterial homogenization process. In this process, ring resonators are represented by point magnetic dipoles. The complex effective permeability of the metamaterial is calculated.

The random distribution of the resonance frequencies of particular current loops - BC-SRRs - significantly widens the resonance frequency band of the effective permeability of the metamaterial. The magnitude of the real and imaginary parts of the effective permeability decrease, the frequency band of real negative part is narrowed, and at higher standard deviation values for the resonance frequencies this band disappears. The procedure was verified experimentally by measuring the scattering parameters of the metamaterial block in the rectangular waveguide. This data can be well compared with the parameters calculated by the CST Microwave Studio, which uses complex permeability determined by the homogenization procedure.

Finally, the presented homogenization procedure was used to determine the limit in which the dispersion of the resonance frequencies of particular resonators can be varied to obtain a working lens designed for improving the quality of the images of the 1.5 T MRI system working at 63.87 MHz. The homogenization procedure was modified here for an analysis of a structure with cubic cells, where the resonant elements are located on particular faces of the cubes. At the same time, the procedure was verified by proving that the analyzed structure shows an isotropic response.