Characteristic Modes
Part I: Introduction

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Seminar
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Characteristic Modes

Conventionally, characteristic modes $\mathbf{I}_n$ are defined as

$$\mathbf{XI}_n = \lambda_n \mathbf{R}_n,$$

in which $\mathbf{Z} = \mathbf{R} + j\mathbf{X}$ is the impedance matrix.
Conventionally, characteristic modes $I_n$ are defined as

$$X I_n = \lambda_n R I_n,$$

in which $Z = R + jX$ is the impedance matrix.

However, who knows:

- What is impedance matrix and how to get it?
- What the hell are the characteristic modes?
- Why are they of our interest?
Characteristic Modes

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Therefore, …

… the characteristic mode theory is to be systematically derived.
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However, who knows:

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- What the hell are the characteristic modes?
- Why are they of our interest?

Therefore, …

… the characteristic mode theory is to be systematically derived.

Disclaimer: There will be equations! Brace yourself and be prepared…
This talk concerns:

- electric currents in vacuum (generalization is, however, straightforward),
- time-harmonic quantities, \( i.e., \mathcal{A}(r, t) = \text{Re} \{ \mathcal{A}(r) \exp(j\omega t) \} \).
Electric Field Integral Equation\(^1\)

\[ \sigma \to \infty \quad \text{(PEC)} \]

Original problem.

Electric Field Integral Equation\(^1\)

\[ \hat{n} \times (E_{S}(r') + E_{i}(r')) = 0, \quad r' \in \Omega \]

Electric Field Integral Equation\textsuperscript{1}

Original problem.
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Equivalent problem.

Electric Field Integral Equation\(^1\)

\[
\hat{n} \times (E_s (r') + E_i (r')) = 0, \quad r' \in \Omega
\]

\[
-k E_s (r') = \sigma \rightarrow \infty \quad \text{(PEC)}
\]

Original problem.

\[
-k E_i (r) = \hat{n} \times \hat{n} \times E_i (r') = Z(J), \quad J = J (r')
\]

Equivalent problem.

Key role of the impedance operator $Z (J)$

$$\hat{n} \times \hat{n} \times E_s (r') = Z (J) = -\hat{n} \times \hat{n} \times (j\omega A + \nabla \varphi).$$

Substituting for Lorenz gauge-calibrated potentials\(^2\) $A$ and $\varphi$ gives

$$Z (J) = jkZ_0 \int_{\Omega} G (r, r') \cdot J (r') \, dS$$

---

Electric Field Integral Equation – Problem Formalization

Key role of the impedance operator $\mathcal{Z}(J)$

$$\hat{n} \times \hat{n} \times E_s(r') = \mathcal{Z}(J) = -\hat{n} \times \hat{n} \times (j\omega A + \nabla \varphi).$$

Substituting for Lorenz gauge-calibrated potentials$^2$ $A$ and $\varphi$ gives

$$\mathcal{Z}(J) = jkZ_0 \int_{\Omega} G(r, r') \cdot J(r') \, dS = jkZ_0 \int_{\Omega} \left(1 + \frac{1}{k^2} \nabla \nabla\right) \cdot J(r') \frac{e^{-jk|r'-r|}}{4\pi |r' - r|} \, dS,$$

- Impedance operator $\mathcal{Z}$ is linear, symmetric (reciprocal, thus non-Hermitian).
- Alternative formulation MFIE$^3$, common extension towards CFIE$^3$.

---

only canonical bodies can typically be evaluated analytically. Problem $-\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{E}_i (r') = \mathcal{Z} (\mathbf{J})$ has to be solved numerically!

Equivalent problem.

---

Dicretization of the Problem

Only canonical bodies can typically be evaluated analytically. Problem $-\mathbf{n} \times \mathbf{n} \times \mathbf{E}_i (r') = \mathcal{Z} (J)$ has to be solved numerically!

- Discretization\(^4\) $\Omega \rightarrow \Omega_T$ is needed (nontrivial task!)

Representation of the Operator (Recap.)

Engineers like linear systems

\[ \mathcal{L}(f) = h. \]

- Typically unsolvable for \( f \) in the present state (how to invert \( \mathcal{L} \)?)

Linear system with input \( f \) and output \( g \).
Representation of the Operator (Recap.)

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- Typically unsolvable for \( f \) in the present state (how to invert \( \mathcal{L} \)?)

Representation in a basis \( \{\psi_n\} \) and linearity of operator \( \mathcal{L} \) readily gives

\[
\sum_{n=1}^{N} I_n \mathcal{L}(\psi_n) = h.
\]

- One equation for \( N \) unknowns → still unsolvable.

---

Using proper inner product $\langle \cdot, \cdot \rangle$ and $N$ tests from left, we get

$$\sum_{n=1}^{N} I_n \langle \chi_n, \mathcal{L} (\psi_n) \rangle = \langle \chi_n, h \rangle,$$

i.e., in matrix form the method of moments\(^5\) relation reads

$$L I = H.$$

---

Algebraic Solution – Method of Moments

Piecewise basis functions\(^6\)

\[
\psi_n(r) = \frac{l_n}{2A_n} \rho^\pm(r)
\]

RWG basis function \(\psi_n\).

Piecewise basis functions\(^6\)

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\psi_n (r) = \frac{l_n}{2A_n} \rho^\pm (r)
\]

are applied to approximate \(J (r)\) as

\[
J (r) \approx \sum_{n=1}^{N} I_n \psi_n (r).
\]

---


RWG basis function \(\psi_n\).
Piecewise basis functions\textsuperscript{6}\\  
\[ \psi_n(r) = \frac{l_n}{2A_n^\pm} \rho^\pm(r) \]

are applied to approximate \( J(r) \) as\\  
\[ J(r) \approx \sum_{n=1}^{N} I_n \psi_n(r). \]

Galerkin testing\textsuperscript{7}, \textit{i.e.}, \{\chi_n\} = \{\psi_n\}, is performed as\\  
\[ \int_{\Omega} \psi \cdot Z(\psi) \, dS = \langle \psi, Z(\psi) \rangle \equiv Z = [Z_{pq}] \in \mathbb{C}^{N \times N}. \]

---
\textsuperscript{7}P. M. Morse and H. Feshbach, Methods of Theoretical Physics. McGraw-Hill, 1953
From Impedance Operator $\mathcal{Z}$ to Impedance Matrix $\mathbf{Z}$

The impedance matrix $\mathbf{Z}$ reads

$$Z_{pq} = \int_{\Omega} \psi_p \cdot \mathcal{Z} (\psi_q) \, dS = jk Z_0 \int_{\Omega} \int_{\Omega} \psi_p (r_1) \cdot \mathbf{G}(r_1, r_2) \cdot \psi_q (r_2) \, dS_1 \, dS_2.$$
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We say\(^8\): “Matrix $\mathbf{Z}$ is the impedance operator $\mathcal{Z}$ represented in $\{\Psi_n\}$ basis.”

From Impedance Operator $\mathcal{Z}$ to Impedance Matrix $Z$

The impedance matrix $Z$ reads

$$Z_{pq} = \int_{\Omega} \psi_p \cdot \mathcal{Z}(\psi_q) \, dS = jkZ_0 \int_{\Omega} \int_{\Omega} \psi_p(\mathbf{r}_1) \cdot \mathbf{G}(\mathbf{r}_1, \mathbf{r}_2) \cdot \psi_q(\mathbf{r}_2) \, dS_1 \, dS_2.$$ 

- We say\textsuperscript{8}: “Matrix $Z$ is the impedance operator $\mathcal{Z}$ represented in $\{\psi_n\}$ basis.”
- Matrix $Z$ can be calculated, \textit{e.g.}, in AToM\textsuperscript{9} (plenty of numerical techniques and tricks should/can be used\textsuperscript{10}).

\textsuperscript{9}(2017). Antenna Toolbox for MATLAB (AToM), Czech Technical University in Prague, [Online]. Available: www.antennatoolbox.com
From Impedance Operator $\mathcal{Z}$ to Impedance Matrix $\mathbf{Z}$

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- Generally$^{11}$, impedance matrix $\mathbf{Z}$ inherits properties of impedance operator $\mathcal{Z}$.
  - Symmetric, complex-valued.

---


From Impedance Operator $\mathcal{Z}$ to Impedance Matrix $\mathbf{Z}$

The impedance matrix $\mathbf{Z}$ reads

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- Matrix $\mathbf{Z}$ can be calculated, \textit{e.g.}, in AToM\textsuperscript{9} (plenty of numerical techniques and tricks should/can be used\textsuperscript{10}).
- Generally\textsuperscript{11}, impedance matrix $\mathbf{Z}$ inherits properties of impedance operator $\mathcal{Z}$.
  - Symmetric, complex-valued.
- Matrix $\mathbf{Z}$ completely describe the scattering properties of radiator $\Omega_T$.

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\textsuperscript{9} (2017). Antenna Toolbox for MATLAB (AToM), Czech Technical University in Prague, [Online]. Available: \url{www.antennatoolbox.com}
### Two Hilbert Space Representations\textsuperscript{12}

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<td>$\mathcal{Z} = \langle r, \mathcal{Z}(r) \rangle$</td>
<td>$\mathcal{Z} = \langle \psi, \mathcal{Z}(\psi) \rangle$</td>
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<tr>
<td>bilinear form (for $\mathcal{Z}$)</td>
<td>$p = \langle J, \mathcal{Z}(J) \rangle$</td>
<td>$p \approx I^H ZI$</td>
</tr>
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</table>

$$\langle f, g \rangle = \int_{\Omega} f^*(x) \cdot g(x) \, dx, \quad A^H = (A^T)^*$$

Example: Complex Power Balance

Quantity being important in the following.
Example: Complex Power Balance

Quantity being important in the following.

- Continuous form\(^\text{13}\) using operator \(\mathcal{Z}\)

\[
-\frac{1}{2} \int_{\Omega} J^* \cdot E_s \, dS = \frac{1}{2} \int_{\Omega} J^* \cdot \mathcal{Z} (J) \, dS = P_{\text{rad}} + 2j\omega (W_m - W_e).
\]

---

Example: Complex Power Balance

Quantity being important in the following.

- **Continuous form**\(^\text{\ref{13}}\) using operator \(Z\)

\[
-\frac{1}{2} \int_{\Omega} \mathbf{J}^* \cdot \mathbf{E}_s \, dS = \frac{1}{2} \int_{\Omega} \mathbf{J}^* \cdot Z(\mathbf{J}) \, dS = P_{\text{rad}} + 2j\omega (W_m - W_e). 
\]

- **Algebraic form**\(^\text{\ref{14}}\) using matrix \(Z\)

\[
\frac{1}{2} \int_{\Omega} \mathbf{J}^* \cdot Z(\mathbf{J}) \, dS \approx \frac{1}{2} \mathbf{I}^H \mathbf{Z} \mathbf{I} = P_{\text{rad}} + 2j\omega (W_m - W_e). 
\]


Motivation

Describe behavior of a scatterer $\Omega$ without feeding considered.
Definition of Characteristic Modes

Diagonalization of the Impedance Operator/Matrix

Motivation

Describe behavior of a scatterer $\Omega$ without feeding considered.

Diagonalization of the impedance matrix $Z$:

$$
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} & \cdots & Z_{1N} \\
Z_{21} & Z_{22} & Z_{23} & \cdots & Z_{2N} \\
Z_{31} & Z_{32} & Z_{33} & \cdots & Z_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Z_{N1} & Z_{N2} & Z_{N3} & \cdots & Z_{NN}
\end{bmatrix}
$$

Impedance matrix $Z$.

$$
\begin{bmatrix}
\nu_1 & 0 & 0 & \cdots & 0 \\
0 & \nu_2 & 0 & \cdots & 0 \\
0 & 0 & \nu_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \nu_N
\end{bmatrix}
$$

Yet-unknown diagonalization of impedance matrix $Z$. 
Impedance Operator Represented in Spherical Harmonics

Example: Let us represent the impedance operator $\mathcal{Z}$ in a basis\(^{15}\) of spherical harmonics $\{J_{sh}^m(\vartheta, \varphi, a)\} \in \mathbb{R}$

$$\langle J^m_{sh}, \mathcal{Z}(J^m_{sh}) \rangle = \int_{\Omega} J^m_{sh} \cdot \mathcal{Z}(J^m_{sh}) \, dS,$$

Spherical shell of radius $a$.

---

\(^{15}\)This can be understood as solving method of moments analytically with spherical harmonics as basis functions.

Definition of Characteristic Modes

Impedance Operator Represented in Spherical Harmonics

Example: Let us represent the impedance operator $\mathcal{Z}$ in a basis\textsuperscript{15} of spherical harmonics $\{J^{\text{sh}}_n(\vartheta, \varphi, a)\} \in \mathbb{R}$

$$\langle J^\text{sh}_m, \mathcal{Z}(J^\text{sh}_n) \rangle = \int _\Omega J^\text{sh}_m \cdot \mathcal{Z}(J^\text{sh}_n) \, \text{d}S,$$

This representation gives diagonal matrix\textsuperscript{16}, i.e.,

$$\langle J^\text{sh}_m, \mathcal{Z}(J^\text{sh}_n) \rangle = 2 \left( P^\text{sh}_{\text{rad}, n} + 2j \omega (W^\text{sh}_{m,n} - W^\text{sh}_{e,n}) \right) \delta_{mn},$$

$\delta_{mn} = 1 \iff m = n \quad \land \quad \delta_{mn} = 0 \iff m \neq n.$

---

\textsuperscript{15} This can be understood as solving method of moments analytically with spherical harmonics as basis functions.

Characteristic Modes of Spherical Shell

Formula for spherical shell normalized to unitary radiated power (no units)

\[
\frac{\langle J_{sh}^m, Z (J_{sh}^n) \rangle}{\langle J_{sh}^m, R (J_{sh}^n) \rangle} = \left( 1 + j \frac{2\omega (W_{sh}^{m,n} - W_{e,n})}{P_{rad,n}} \right) \delta_{mn},
\]

where \( Z = R + j\lambda \)

\[\text{ǎCapek, M. Characteristic Modes – Part I: Introduction 15 / 39}\]
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\]

where \( Z = R + j\lambda \) or, alternatively (without problems due to division by zero),

\[
\langle J_{sh}^m, R (J_{sh}^n) + j\lambda (J_{sh}^n) \rangle = \left( 1 + j\lambda_{sh}^n \right) \langle J_{sh}^m, R (J_{sh}^n) \rangle \delta_{mn}.
\]
Characteristic Modes of Spherical Shell

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where \( Z = R + jX \) or, alternatively (without problems due to division by zero),

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\]

Linearity of the impedance operator allows to write

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Characteristic Modes of Spherical Shell

Formula for spherical shell normalized to unitary radiated power (no units)

$$\frac{\langle J_{m,}^{\text{sh}}, Z(J_{n}^{\text{sh}}) \rangle}{\langle J_{m,}^{\text{sh}}, R(J_{n}^{\text{sh}}) \rangle} = \left(1 + j \frac{2 \omega (W_{m,n}^{\text{sh}} - W_{e,n}^{\text{sh}})}{P_{\text{rad},n}^{\text{sh}}} \right) \delta_{mn},$$

where $Z = R + j \chi$ or, alternatively (without problems due to division by zero),

$$\langle J_{m,}^{\text{sh}}, R(J_{n}^{\text{sh}}) + j \chi(J_{n}^{\text{sh}}) \rangle = \left(1 + j \lambda_{n}^{\text{sh}}\right) \langle J_{m,}^{\text{sh}}, R(J_{n}^{\text{sh}}) \rangle \delta_{mn}.$$

Linearity of the impedance operator allows to write

$$\langle J_{m,}^{\text{sh}}, \chi(J_{n}^{\text{sh}}) \rangle = \lambda_{n}^{\text{sh}} \langle J_{m,}^{\text{sh}}, R(J_{n}^{\text{sh}}) \rangle \delta_{mn},$$

which is solved for all $n \in \{1, \ldots, \infty\}$ via generalized eigenvalue problem (GEP)

$$\chi(J_{n}^{\text{sh}}) = \lambda_{n}^{\text{sh}} R(J_{n}^{\text{sh}}).$$
Characteristic modes for arbitrarily shaped body are defined as GEP

\[ \chi(J_n) = \lambda_n R(J_n). \]
Characteristic modes for arbitrarily shaped body are defined as GEP

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Algebraic form\(^{17}\) is commonly used instead

\[ X I_n = \lambda_n R I_n, \]

with \( Z = R + jX \) being the impedance matrix and \( I_n \in \mathbb{R}^{N \times 1} \) being expansion coefficients.

\(^{17}\)Only six canonical bodies can, in principle, be solved analytically.
Characteristic modes for arbitrarily shaped body are defined as GEP

\[ \mathbf{\chi}(J_n) = \lambda_n \mathbf{R}(J_n). \]

Algebraic form\(^{17}\) is commonly used instead

\[ \mathbf{X}\mathbf{I}_n = \lambda_n \mathbf{R}\mathbf{I}_n, \]

with \( \mathbf{Z} = \mathbf{R} + j\mathbf{X} \) being the impedance matrix and \( \mathbf{I}_n \in \mathbb{R}^{N \times 1} \) being expansion coefficients.

We know that GEP\(^{18}\) is capable to diagonalize both \( \mathbf{R} \) and \( \mathbf{X} \) operators\(^{19}\).

- Behavior solely described by the impedance operator/matrix.
- No feeding present (neither \( \mathbf{E}_i \), nor \( \mathbf{V} \))!

---

\(^{17}\)Only six canonical bodies can, in principle, be solved analytically.


\(^{19}\)Generally, only two operators can simultaneously be diagonalized. Separable bodies are exceptional!
1948 First mention of diagonalization of the scattering operator by Montgomery et al.\textsuperscript{20}

\textsuperscript{23} The literature will be closely reviewed later.

Historical Overview\textsuperscript{23}

1948  First mention of diagonalization of the scattering operator by Montgomery \textit{et al.}\textsuperscript{20}

1968  Rigorously introduced by Garbacz\textsuperscript{21} as field/current solutions $E_n/J_n$ with orthogonal far-fields and radiating unitary power.

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1971 Generalized by Harrington and Mautz\textsuperscript{22} for antenna problem using impedance matrix $Z$ as $ZI_n = \nu_n MI_n$, $\nu_n \equiv 1 + j\lambda_n$, $M \equiv R$.

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Historical Overview

1948 First mention of diagonalization of the scattering operator by Montgomery et al.\(^{20}\)

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- analytical form

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- algebraic form

---

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Characteristic Numbers $\lambda_n$

Rayleigh quotient\(^{24}\) is defined as

$$
\lambda_n = \frac{\int_{\Omega} J_n^* \cdot \mathcal{X}(J_n) \, dS}{\int_{\Omega} J_n^* \cdot \mathcal{R}(J_n) \, dS} = \frac{2\omega (W_{m,n} - W_{e,n})}{P_{\text{rad},n}} \approx \frac{I_n^H X I_n}{I_n^H R I_n}.
$$

Properties of Characteristic Modes

Characteristic Numbers $\lambda_n$

Rayleigh quotient is defined as

$$\lambda_n = \frac{\int_{\Omega} J_n^* \cdot \mathcal{X}(J_n) \ dS}{\int_{\Omega} J_n^* \cdot \mathcal{R}(J_n) \ dS} = \frac{2\omega (W_{m,n} - W_{e,n})}{P_{\text{rad},n}} \approx \frac{I_n^H X I_n}{I_n^H R I_n}.$$ 

Notice $P_{\text{rad},n} > 0 \Rightarrow R > 0$ is required.

Eigenvalues $\lambda_n$ represent the stationary points.

---


Physical Meaning of Characteristic Numbers $\lambda_n$

Characteristic modes can be classified as:

- $W_{m,n} > W_{e,n} \Rightarrow \lambda_n > 0$ mode is of inductive nature,
- $W_{m,n} < W_{e,n} \Rightarrow \lambda_n < 0$ mode is of capacitive nature,
- $W_{m,n} = W_{e,n} \Rightarrow \lambda_n = 0$ mode is in resonance\(^{26}\).

▶ To get current to the resonance, let us combine modes with $\lambda_n < 0$ and $\lambda_n > 0$.

\(^{26}\)Resonance of a modal current impressed in vacuum is doubtful. It cannot be excited independently.
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Knowledge in group theory\(^{27}\) gives understanding of

- degeneracies,
- crossings,
- crossing avoidances.

---

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Properties of Characteristic Modes

Cardinality of a Set of Characteristic Modes

For a radiator $\Omega \ni r'$ of finite extent:

\[ J_{n} = \sum \alpha_{n} J_{n} \]

Mapping between current densities and their complex power ratios.

Set of characteristic modes is infinite, but countable.

Set of all currents has higher cardinality (uncountable).
Cardinality of a Set of Characteristic Modes

For a radiator $\Omega \ni r'$ of finite extent:

Characteristic modes can freely be combined as

$$J = \sum_{n} \alpha_n J_n.$$
Cardinality of a Set of Characteristic Modes

For a radiator $\Omega \ni r'$ of finite extent:

$J = \sum_{n} \alpha_n J_n.$

- Set of characteristic modes is infinite, but countable.
- Set of all currents has higher cardinality (uncountable).
Characteristic Numbers $\lambda_n$ for Spherical Shell

$W_m < W_e$

$\lambda_n$ vs $ka$

TM modes
Characteristic Numbers $\lambda_n$ for Spherical Shell
Characteristic angles \( \delta_n \) scale the dynamics of \( \lambda_n \in (-\infty, \infty) \)

\[
\delta_n = 180^\circ \left( 1 - \frac{1}{\pi} \arctan (\lambda_n) \right).
\]

to \( \delta_n \in (90^\circ, 270^\circ) \).

---

Characteristic Eigenangles $\delta_n$

Characteristic angles $\delta_n$ scale the dynamics of $\lambda_n \in (-\infty, \infty)$

$$
\delta_n = 180^\circ \left( 1 - \frac{1}{\pi} \arctan (\lambda_n) \right).
$$

to $\delta_n \in (90^\circ, 270^\circ)$.

Similarly as for characteristic numbers:

$\lambda_n > 0 \Rightarrow \delta_n < 180^\circ$ mode is of inductive nature,

$\lambda_n < 0 \Rightarrow \delta_n > 180^\circ$ mode is of capacitive nature,

$\lambda_n = 0 \Rightarrow \delta_n = 180^\circ$ mode is in resonance.

---

Characteristic Eigenangles $\delta_n$ for Spherical Shell

$\delta_n$ values for TM modes are given as:
- $\delta_n = 0$ for $0 < ka < 0.5$
- $\delta_n = 180$ for $4 < ka < 5$

$W_m < W_e$ for $ka$ values between 0.5 and 4.
Characteristic Eigenangles $\delta_n$ for Spherical Shell

\[ \begin{align*}
\delta_n & \quad 0 \quad 0.5 \quad 1 \quad 1.5 \quad 2 \quad 2.5 \quad 3 \quad 3.5 \quad 4 \quad 4.5 \quad 5 \\
W_m & < W_e \\
W_m & > W_e \\
\end{align*} \]
Spherical Shell Solved Numerically

While simplest canonical body, spherical shell has plenty of potential issues, e.g.,

- degenerate eigenspace\(^{29}\), \(D(l) = 2l + 1\),
- conformity of spherical surface with commonly used basis functions,
- internal resonances\(^{30}\) \((\lambda_n \to \infty)\),
- computationally demanding evaluation\(^{31}\).

Spherical shell made of 194 triangles.


\(^{31}\) Having no junctions, spherical shell (and other closed objects) has highest possible ratio between number of basis functions (unknowns) and triangles (3/2).
Dominant capacitive characteristic mode (spherical harmonic $\text{TM}_{10}$).

Dominant inductive characteristic mode (spherical harmonic $\text{TE}_{10}$).
Orthogonality relations\textsuperscript{32}

\[
\frac{1}{2} I_m^H Z I_n = (1 + j\lambda_n) \delta_{mn},
\]

\[
\frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} \int_0^{2\pi} \int_0^\pi F_m^* \cdot F_n \sin \vartheta \, d\vartheta \, d\varphi = \delta_{mn},
\]

i.e., orthogonalization of modal complex power and modal far-fields.

Summation of Characteristic Modes

Summation formula

\[ J = \sum_n \alpha_n J_n \]
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derived using linearity of the impedance operator and orthogonality of characteristic modes as

\[ J = \sum_n \frac{\langle J_n, E_i \rangle}{\langle J_n, Z(J_n) \rangle} J_n = \sum_n \frac{V^i_n}{1 + j\lambda_n} J_n \]
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with \( V^i_n \) being modal excitation coefficient\(^{33}\) and \( M_n = 1/|1 + j\lambda_n| \) being modal significance coefficient\(^{34}\).

▶ Connection between “external” and “modal” worlds.

Bonus: On the Inversion of $\mathcal{Z}$ operator

Summation formula slightly rearranged (Dirac notation used, i.e., $\mathcal{L}|f\rangle = |g\rangle$

$$|J\rangle = \sum_{n} \frac{\langle J_{n}|E_{i}\rangle}{\langle J_{n}|\mathcal{Z}|J_{n}\rangle} |J_{n}\rangle$$
**Bonus: On the Inversion of $\mathcal{Z}$ operator**

Summation formula slightly rearranged (Dirac notation used, i.e., $\mathcal{L}|f\rangle = |g\rangle$)

$$|J\rangle = \sum_n \frac{\langle J_n|E_i\rangle}{\langle J_n|\mathcal{Z}|J_n\rangle}|J_n\rangle$$

Let’s do some magic…

$$|J\rangle = \sum_n \frac{|J_n\rangle\langle J_n|}{\langle J_n|\mathcal{Z}|J_n\rangle} - \hat{n} \times \hat{n} \times E_i\}$$

and compare with the defining formula $|- \hat{n} \times \hat{n} \times E_i\rangle = \mathcal{Z}|J\rangle$. 

▶

But, do not even try to calculate!
Bonus: On the Inversion of $\mathcal{Z}$ operator

Summation formula slightly rearranged (Dirac notation used, \textit{i.e.}, $\mathcal{L} |f\rangle = |g\rangle$)

$$|J\rangle = \sum_n \frac{\langle J_n | E_i \rangle}{\langle J_n | \mathcal{Z} | J_n \rangle} |J_n\rangle$$

Let’s do some magic…

$$|J\rangle = \sum_n \frac{|J_n\rangle \langle J_n|}{\langle J_n | \mathcal{Z} | J_n \rangle} |\hat{n} \times \hat{n} \times E_i\rangle$$

and compare with the defining formula $| - \hat{n} \times \hat{n} \times E_i\rangle = \mathcal{Z} |J\rangle$.

We get

$$\mathcal{Z}^{-1} \equiv \sum_n \frac{|J_n\rangle \langle J_n|}{\langle J_n | \mathcal{Z} | J_n \rangle}.$$  

▶ But, do not even try to calculate!
Good theory needs time…

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Notes on Characteristic Modes

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3. Singular expansion method\(^{38}\), i.e., \(\mathbf{Z}\mathbf{I}_n = 0\).

---


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4. Other representations\textsuperscript{39} applicable, \textit{i.e.}, $\hat{\mathbf{Z}} = [\langle \mathbf{f}_m, \mathbf{Z} \mathbf{f}_n \rangle]$.


\textsuperscript{39} A. J. King, “Characteristic mode theory for closely spaced dipole arrays”, PhD thesis, University of Illinois at Urbana-Champaign, 2015
Notes on Characteristic Modes

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5. Other bases\(^{40}\) exist, \(\text{i.e.}, \mathbf{A} \mathbf{I}_n = \xi_n \mathbf{B} \mathbf{I}_n\).
6. Nowadays implemented in FEKO, CST-MWS, WIPL-D, CEM One, HFSS.


Activities at the Department

Topics Recently Solved at the Department

▶ Analytical properties of characteristic modes\(^{41}\),
▶ implementation of characteristic modes\(^{42}\),
▶ benchmarks of commercial and in-house solvers\(^{43}\),
▶ modal Q-factor for antennas\(^{44}\),
▶ radiation efficiency of characteristic modes\(^{45}\),
▶ minimization of Q-factor using characteristic modes\(^{46}\).


Ongoing Research at the Department

- Improvement of characteristic modes decomposition\textsuperscript{47}.
- tracking of modal data\textsuperscript{48},
- group theory (symmetries) for tracking and problem reducing\textsuperscript{49},
- characteristic modes for MLFMA\textsuperscript{50},
- interpolation using differentiated GEP,
- characteristic modes for arrays\textsuperscript{51}.

AToM – Antenna Toolbox for MATLAB

AToM developed at the department between 2014 and 2018.

- Capable to calculate matrix $\mathbf{Z}$ (and many other matrices),
- capable to calculate the CMs, their tracking, post-processing.

Visit antennatoolbox.com
Special Interested Group

- Established by prof. Lau, Lund University,
- 77 groups worldwide, CTU/Elmag is an active member!

Visit characteristicmodes.org

Benchmarking of the CMs Decomposition

Visit elmag.org/CMbenchmark

$\log_{10} |\lambda_n|$

$\begin{array}{cccccccc}
TM/TE mode order & 63 & 48 & 35 & 24 & 15 & 8 & 3 & 3 & 8 & 15 & 24 & 35 & 48 & 63 \\
\end{array}$

Characteristic mode analysis

From Wikipedia, the free encyclopedia

**Characteristic modes (CM)** form a set of functions which, under specific boundary conditions, diagonalizes operator relating field and induced sources.

Under certain conditions, the set of the CM is unique and complete (at least theoretically) and thereby capable of describing the behavior of a studied object in full.

This article deals with characteristic mode decomposition in electromagnetics, a domain in which the CM theory has originally been proposed.

### Background

CM decomposition was originally introduced as set of modes diagonalizing a scattering matrix. The theory has, subsequently, been generalized by Harrington and Mautz for antennas. Harrington, Mautz and their students also successively developed several other extensions of the theory.

Even though some precursors were published back in the late 1940s, the full potential of CM has remained unrecognized for an additional 40 years. The capabilities of CM were revisited in 2007 and, since then, interest in CM has dramatically increased. The subsequent boom of CM theory is reflected by

### Definition

For the simplicity, only the original form of the CM — formulated for perfectly electrically conducting (PEC) bodies in free space — will be treated in this introduction.
European School of Antennas

Course on

Characteristic Modes: Theory and Applications

Aimed at postgraduate research students and industrial engineers who want to acquire deep insight into the theory and applications of characteristic modes.

Visit esoa.webs.upv.es
Characteristic modes decomposition

\[ \mathbf{XI}_n = \lambda_n \mathbf{RI}_n \]

- diagonalizes impedance matrix \( \mathbf{Z} \),
- constitutes entire domain basis \( \{ \mathbf{I}_n \} \),
- generates orthogonal far-fields \( \{ \mathbf{F}_n \} \),
- allows compact representation of the radiator \( \Omega_T \).

\[ \sigma \to \infty \quad \text{(PEC)} \]

\[ \epsilon_0, \mu_0 \]

\[ \mathbf{J}(r') \]

\[ \Omega_T \]
Questions?

For a complete PDF presentation see capek.elmag.org

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