Some Numerical Aspects of Characteristic Mode Decomposition

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The 10th European Conference on Antennas and Propagation
Davos, Switzerland
April 11, 2016
In this talk:

- electric currents in vacuum,
- time-harmonic quantities, i.e., $A(r, t) = \sqrt{2} \text{Re} \{A(r) \exp(j\omega t)\}$ are considered.
Characteristic mode (CM) decomposition as proposed by Harrington\(^1\)

\[ \mathbf{XI}_n = \lambda_n \mathbf{RI}_n \]  

\[ \lambda_n = \frac{\mathbf{I}_n^H \mathbf{XI}_n}{\mathbf{I}_n^H \mathbf{RI}_n} \]  

\( n, m \in \{1, \ldots, N\} \)

\[ \mathbf{Z} = \mathbf{R} + j\mathbf{X} = \begin{bmatrix} Z_{mn} \end{bmatrix}, \quad Z_{mn} = \langle \mathbf{f}_m, \mathbf{Z}\mathbf{f}_n \rangle \]

\[ \mathbf{J} \approx \sum_m I_m \mathbf{f}_m, \quad \mathbf{J} \approx \sum_n \alpha_n \mathbf{J}_n, \quad \mathbf{I} \approx \sum_n \alpha_n \mathbf{I}_n, \quad \mathbf{I}_n = \begin{bmatrix} I_{mn} \end{bmatrix} \]

\[ \mathbf{R} \succ 0 \]

\[ \mathbf{A}^H = \left( \mathbf{A}^T \right)^*, \quad \langle \mathbf{a}, \mathbf{b} \rangle = \int_{\Omega} \mathbf{a}^* \cdot \mathbf{b} \, dV \]

Characteristic Mode (CM) decomposition

Analytical form of CM decomposition\(^2\) \((n \in \mathbb{N}^+)\)

\[
\chi J_n = \lambda_n R J_n \tag{3}
\]

\[
\lambda_n = \frac{\langle J_n, \chi J_n \rangle}{\langle J_n, R J_n \rangle} \tag{4}
\]

\[\blacktriangleright\] difference between (1) and (3) seems to be only formal,

\[\blacktriangleright\] but it is not…

\[\mathcal{S} = \mathcal{R} + j\chi = -j\omega n_0 \times \left( \mathcal{A} + \frac{1}{k^2} \nabla \nabla \cdot \mathcal{A} \right)\]

\[
\mathcal{A}(\ast) = \frac{\mu}{4\pi} \int_V (\ast) \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \, dV', \quad \mathcal{A}(\mathbf{r}) = \mathcal{A} J(\mathbf{r}')
\]

Utilization of the analytical functional

Why solution for $J_n$ is of interest?

Analytical form of the CM decomposition

- diagonalizes the impedance operator $\mathcal{Z}$
- and has its...

\[ \mathcal{X} J_n = \lambda_n \mathcal{R} J_n \]
\[ \lambda_n = \frac{\langle J_n, \mathcal{X} J_n \rangle}{\langle J_n, \mathcal{R} J_n \rangle} \]

Potentialities:

- it reveals exact structure of the CM decomposition
- it can be utilized for several theoretical studies
  - (see later)
- it can be utilized for benchmarking
  - (see later)

Limitations:

- it can be evaluated only for a small class of radiating bodies\(^3\)
  - cannot be used for calculations involving arbitrarily shaped bodies

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Utilization of the analytical functional

Analytical form of Rayleigh quotient

Analytical structure of Rayleigh quotient for CM:

\[
\begin{align*}
\lambda_n &= -\frac{\text{Re} \int_{\Omega} (A \cdot J_n^* - \phi \rho_n^*) \, dV}{\text{Im} \int_{\Omega} (A \cdot J_n^* - \phi \rho_n^*) \, dV} = \frac{\int_{\Omega_1} \int_{\Omega_2} \mathcal{J} J_n \frac{\cos(kR)}{R} \, dV_2 \, dV_1}{\int_{\Omega_1} \int_{\Omega_2} \mathcal{J} J_n \frac{\sin(kR)}{R} \, dV_2 \, dV_1} \\
\end{align*}
\]

(5)

What about \(\lambda_n\) when different \(J_n\) than characteristic mode is substituted?

\[
\mathcal{J}(\star) = k^2 (\star) \cdot (\star)^* - \nabla_1 \cdot (\star) \nabla_2 \cdot (\star)^* \\
R = |r_1 - r_2|, \quad r_1 \in V_1, \quad r_2 \in V_2
\]
Utilization of the analytical functional

Utilization of the analytical operator

Example of a dipole

we can test other candidates

Current density

\[ \tilde{J}_n(z) = z_0 \frac{1}{w} \delta(y) \sin \left( \frac{\pi n z}{L} \right), \quad x \in \left( -\frac{w}{2}, \frac{w}{2} \right), \quad z \in \left( -\frac{L}{2}, \frac{L}{2} \right) \]

and charge density

\[ z_0 \cdot \frac{\partial \tilde{J}_n(z)}{\partial z} = \frac{1}{w} \frac{\pi n}{L} \delta(y) \cos \left( \frac{\pi n z}{L} \right) \]

can be substituted into the functional → radiation quotient \( \kappa_n \).

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Utilization of the analytical operator

Example of a dipole – Eigenvalues

▶ reduced kernel\(^5\) with wire-strip modification\(^6\) used

Characteristic numbers \(\lambda_n\) and radiation quotients \(\kappa_n\) for a dipole, \(n \in \{1, 2, 3\}\).

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Utilization of the analytical functional

Utilization of the analytical operator
Example of a dipole – Eigenvectors

- notice similarity between $J_n$ and $\tilde{J}_n$

Characteristic modes $J_n$ and analytical currents $\tilde{J}_n$, $n \in \{1, 2, 3\}$ in their resonances.
Utilization of the analytical functional

Utilization of the analytical operator

Example of two parallel dipoles

—we can test other candidates

Characteristic numbers $\lambda_1$ and radiation quotients $\kappa_1$ for two parallel dipoles.
Modal quality factor $Q$ from CMs

$$Q = \frac{\omega \tilde{W}_{sto}}{P_r}$$

(6)
Modal quality factor $Q$ from CMs

Let us suppose that\textsuperscript{7}

- the $n$-th CM dominates the solution

Thus\textsuperscript{8}

$$Q = \frac{\omega \tilde{W}_{sto}}{P_r} \approx \frac{\omega}{2} \frac{\partial \lambda_n}{\partial \omega} = \frac{\omega}{2} \frac{\partial}{\partial \omega} I_n^H X I_n$$

\textsuperscript{7}The assumptions are valid in the vicinity of the resonance of the $n$-th mode.

Modal quality factor $Q$ from CMs

Let us suppose that\(^7\)

- the $n$-th CM dominates the solution
- current is stationary, i.e., $\partial I_n/\partial \omega \approx 0$
- frequency variations in $\mathbf{R}$ are negligible as compared to those in $\mathbf{X}$

Thus\(^8\)

$$Q = \frac{\omega \tilde{W}_{sto}}{P_r} \approx \frac{\omega}{2} \frac{\partial \lambda_n}{\partial \omega} = \frac{\omega}{2} \frac{\partial}{\partial \omega} \tilde{\mathbf{X}} \mathbf{I}_n \frac{\mathbf{I}_n^H \mathbf{X} \mathbf{I}_n}{\mathbf{I}_n^H \mathbf{R} \mathbf{I}_n} \approx \frac{\omega}{2} \frac{\mathbf{I}_n^H \frac{\partial \mathbf{X}}{\partial \omega} \mathbf{I}_n}{\mathbf{I}_n^H \mathbf{R} \mathbf{I}_n} = \omega \frac{1}{4} \frac{\mathbf{I}_n^H \frac{\partial \mathbf{X}}{\partial \omega} \mathbf{I}_n}{\frac{1}{2} \mathbf{I}_n^H \mathbf{R} \mathbf{I}_n}. \quad (6)$$

\(^7\)The assumptions are valid in the vicinity of the resonance of the $n$-th mode.

Modified stored energy
Expressions for $J_n$ and $I_n$

Vandenbosch’s formula\textsuperscript{9} can formally be written as:

$$\tilde{W}_\text{sto} = \frac{1}{4} \left\langle J_n, \frac{\partial X}{\partial \omega} J_n \right\rangle$$

(7)

and for used basis and testing functions expressed as\textsuperscript{10}

$$\tilde{W}_\text{sto} \approx \frac{1}{4} I_n^H \frac{\partial X}{\partial \omega} I_n.$$  

(8)


1. Approaches introduced by Harrington (1972), Vandenbosch (2010) and Gustafsson et al. (2014) are formally equal.

2. Considering MoM approach $^{11}$, $\tilde{W}_{\text{sto}}$ can be evaluated analytically by differentiating the MoM kernel $^{12}$.

3. In case that inductive modes $^{13}$ can resonate, they have negative quality factor $Q$ since $Q \propto \frac{\partial \lambda_n}{\partial \omega}$, see (6).

   • Inductive mode on a cylinder (radius $\chi$, height $h$):

   $$\tilde{J}_0 (\varphi, r, z) = \varphi_0 K_0 (z) \delta (r - \chi), \quad z \in \left( -\frac{h}{2}, \frac{h}{2} \right)$$

$^{13}$Those with $W_m \rightarrow \infty$ for $\omega \rightarrow 0$.


Cylinder – Resonance of inductive mode

Negative slope of $\lambda_0$ derivative predestines\textsuperscript{14} negative $\tilde{W}_{\text{sto}}\ldots$

Characteristic numbers $\lambda_0$ (AToM – solid lines, CST – cross markers) and radiation quotients $\kappa_0$ (analytical calculation, dashed lines) for cylinder of radius $\chi$ and height $h$.

Cylinder – Negative modified energy

...which has been confirmed both numerically and analytically\textsuperscript{15}.

Modified energy $\widetilde{W}_{sto}$ for a cylinder of various $\chi/h$ ratio for uniform mode $J_0$.

Numerical issues of CM decomposition

Considerable numerical issues related to the CM decomposition

- Stability of CM decomposition for increasing $N$,
- Positive indefiniteness of $\mathbf{I}_n^H \mathbf{R} \mathbf{I}_n$, i.e., $\mathbf{R} \not\succ 0$,
- Tracking of CMs $\{\mathbf{I}_n\}$,
- Precision of impedance matrix assemblage,
- Precision of GEP algorithms\(^{16}\) (Arnoldi / generalized Schur).

All issues above can somehow be treated, however, how successfully?

Call for an ultimate benchmark.

CMs of a spherical shell

Characteristic modes of a spherical shell can be found analytically\textsuperscript{17}

- properly weighted spherical harmonics\textsuperscript{18}

We can test

- equality of characteristic numbers \(\lambda_n\) (see later)
- equality of characteristic vectors \(J_n\), i.e.,

\[
\epsilon_n = \langle J_n^{\text{SPH}}(r) \cdot J_n^{\text{CM}}(r) \rangle
\]  \hspace{1cm} (9)

- assemblage of inversed impedance matrix (see later)
- minimal quality factor \(Q\) can be found using CMs\textsuperscript{19}


Characteristic values of spherical shell

- spherical TE and TM modes have both $2n + 1$ degeneracy
- modes fall to numerical noise quickly ($n > 4$)

Comparison of characteristic numbers $\lambda_n$ of CM decomposition of spherical shell.
The CM projectors\textsuperscript{20} can easily be derived for expansion coefficients

\[
Z^{-1} = \sum_n \frac{I_n I_n^T}{1 + j\lambda_n}.
\]  

(10)

For spherical shell with spherical harmonics \( J_n \) we get

\[
Z^{-1} = [Z_{mn}]^{-1} = L_0^{-1} \left( \sum_p \frac{\langle f_m, J_p \rangle \langle J_p, f_n \rangle}{1 + j\lambda_p} \right) L_0^{-1} 
\]  

(11)

Since

\[
\sum_q I_q f_q = \sum q \alpha_q J_q,
\]

\[
I_n = L_0^{-1} [\langle f_p, J_n \rangle], \quad L_0 = [\langle f_p, f_q \rangle]
\]

Closer look at $L_0$ operator

Matrix $L_0$ has wide usage

- impedance matrix reconstruction, see (11),
- lossy operator\(^{21}\) (if multiplied by surface resistivity $\rho$)

\[
L = \rho L_0 = [\langle f_m, \rho f_n \rangle], \quad P_{\text{heat}} = \frac{1}{2} \mathbf{I}^H L \mathbf{I}, \quad (12)
\]

- describes geometry and topology of a structure,
- shape optimization without heuristic optimization,
- (maybe) CM “physical preconditioning” as

\[
X \mathbf{I}_n = \xi_n (\mathbf{R} + L) \mathbf{I}_n. \quad (13)
\]

Matrix $L$ is well-defined, symmetric, positive semi-definite and sparse.

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Analytical expression for CMs on a dipole

Having a prolate spheroid ($c > a$), we can proceed in the same way as in the case of a spherical shell:

$$\nabla^2 J_q + k^2 J_q = 0$$  \hspace{1cm} (14)

leads to modal basis (similar as spherical harmonics for sphere)

CMs diagonalize impedance operator as

$$\langle J_q, Z J_p \rangle = (1 + j\lambda_p) \delta_{qp}$$  \hspace{1cm} (15)

(15) can be found using

$$\lambda_q = - \frac{\text{Re} \int (A \cdot J_q^* - \phi \rho_q^*) \, dV}{\text{Im} \int (A \cdot J_q^* - \phi \rho_q^*) \, dV}$$  \hspace{1cm} (16)

A limit of prolate spheroid $a \to 0$ is a wire dipole.
Questions?

For complete PDF presentation see capek.elmag.org

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11.4. 2016, v1.0
Elimination of negative eigenvalues of $\mathbf{R}$ matrix (Matlab code):

```matlab
1 [Rvec, Rnum] = eig(real(Zmatrix));
2 Zmatrix = complex(Rvec * (Rnum .* (1 - (Rnum < 0))) / Rvec, imag(Zmatrix));
```

Operators introduced in Dirac notation (rigorously):

$$\mathcal{Z} = \mathcal{R} + j\mathcal{X} = -j\omega \mathbf{n}_0 \times \left( \mathcal{G} + \frac{1}{k^2} \nabla \nabla \cdot \mathcal{G} \right)$$  (17)

$$\nabla^2 \mathcal{G} + k^2 \mathcal{G} = 1$$  (18)

$$\nabla^2 |\mathbf{J}_n\rangle + k^2 |\mathbf{J}_n\rangle = \xi_n |\mathbf{J}_n\rangle$$  (19)

$$\mathcal{Z} |\mathbf{J}_n\rangle = (1 + j\lambda_n) \mathcal{R} |\mathbf{J}_n\rangle$$  (20)

$$\langle \mathbf{J}_m | \mathcal{Z} | \mathbf{J}_n \rangle = (1 + j\lambda_n) \delta_{mn}$$  (21)

$$\langle \mathbf{r}' | \mathcal{G} | \mathbf{r} \rangle = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}, \quad \mathbf{J}_n (\mathbf{r}) = \langle \mathbf{r} | \mathbf{J}_n \rangle$$  (22)