Numerical Benchmark Based on Characteristic Modes of a Spherical Shell

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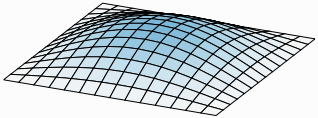
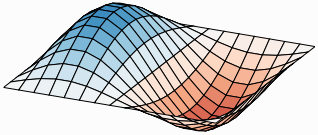
Outline



$\boldsymbol{\mathcal{J}}_1(\boldsymbol{r}, t)$



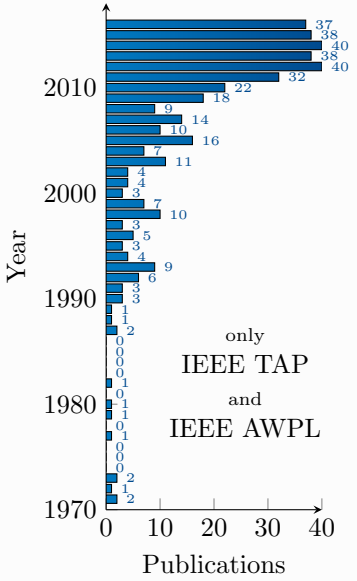
$\boldsymbol{\mathcal{J}}_2(\boldsymbol{r}, t)$

This talk concerns:

- Only PEC objects in vacuum.

Why Benchmarking?

- (Recent) boom of characteristic modes;
- plenty of scientific data have been produced.



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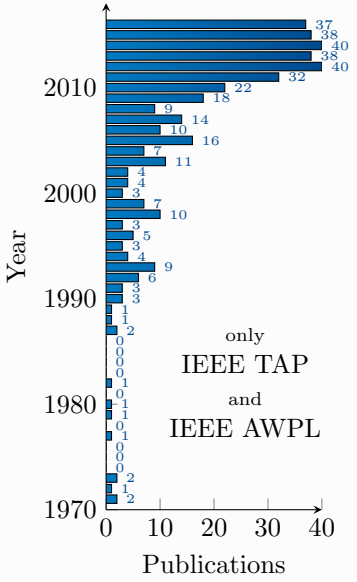
How precisely are our results calculated?



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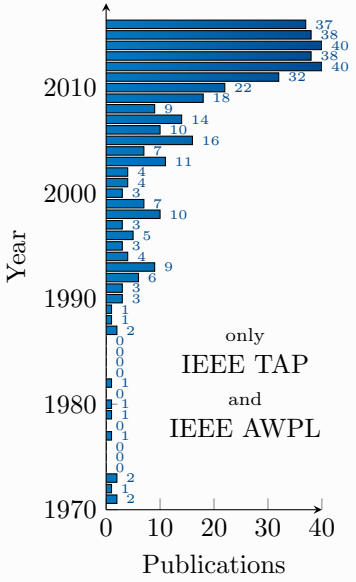
In this presentation:

- analytical solution,
- validation of solvers,
- investigation of convergence.



Separable Systems

$$\nabla^2 f + k^2 f = 0$$

Ideal benchmark = analytically known solution.

Analytical form of CMs potentially exists only in 6 coordinate systems[1] in which vector Helmholtz equations is separable[2].

[1] J. A. Stratton, *Electromagnetic theory*. Wiley – IEEE Press, 2007

[2] W. W. Hansen, "A new type of expansion in radiation problems", *Phys. Rev.*, vol. 47, no. 2, pp. 139–143, 1935. DOI: http://dx.doi.org/10.1103/PhysRev.47.139

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- Li, Kang and Leong[3]: "So far, there are only six coordinate systems in which the scalar Helmholtz equations are separable and solenoidal solutions of the vector Helmholtz equations which are transverse to coordinate surfaces are obtainable."

$$\nabla^2 f + k^2 f = 0$$

$$\begin{aligned} \boldsymbol{L} &= \nabla f \\ \boldsymbol{M} &= \nabla \times \hat{\boldsymbol{a}} f \\ \boldsymbol{N} &= \frac{1}{k} \nabla \times \boldsymbol{M} \\ \boldsymbol{M} &= \boldsymbol{L} \times \hat{\boldsymbol{a}} \\ \boldsymbol{M} &= \frac{1}{k} \nabla \times \boldsymbol{N} \end{aligned}$$

$$\begin{aligned} \nabla \times \boldsymbol{L} &= 0 \\ \nabla \cdot \boldsymbol{M} &= 0 \\ \nabla \cdot \boldsymbol{N} &= 0 \end{aligned}$$

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[2] W. W. Hansen, "A new type of expansion in radiation problems", *Phys. Rev.*, vol. 47, no. 2, pp. 139–143, 1935. DOI: http://dx.doi.org/10.1103/PhysRev.47.139

[3] L.-W. Li, X.-K. Kang, and M.-S. Leong, *Spheroidal wave functions in electromagnetic theory*. New York: Wiley, 2002

Solvable Systems

To get analytical results we should selected one of the following systems:

- rectangular
- spherical
- circular-cylinder
- elliptic-cylinder
- parabolic-cylinder
- conical

Solvable Systems

To get analytical results we should selected one of the following systems:

rectangular	of infinite extent in $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ (plane waves)
spherical	can be done
circular-cylinder	of infinite extent in $\hat{\mathbf{z}}$ (cylindrical waves)
elliptic-cylinder	of infinite extent in $\hat{\mathbf{z}}$ (cylindrical waves)
parabolic-cylinder	of infinite extent in $\hat{\mathbf{z}}$ (cylindrical waves)
conical	for heros only

For other systems, the separability of vector Helmholtz equation is problematic[4].

[4] *E.g.*, for prolate and oblate spheroids, the vector wave functions are not orthogonal to radial spheroidal functions. This induces problems with solving the boundary value problems correctly. Discuss Li, Kang and Leong's book for details.

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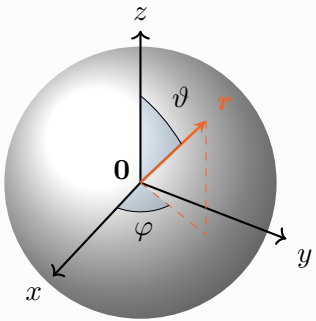
Characteristic Modes of Spherical Shell



Spherical shell of radius a.

Characteristic modes $\boldsymbol{J}_n$ are defined[5] here as

$$\frac{1}{2}\int_{\Omega} \boldsymbol{J}_m^* \cdot \mathcal{Z}\left(\boldsymbol{J}_n\right) \mathrm{d}\Omega = \left(1 + \mathrm{j}\lambda_n\right)\delta_{mn}, \qquad (1)$$

in which

$$\mathcal{Z}\left(\boldsymbol{J}_n\right) = \mathcal{R}\left(\boldsymbol{J}_n\right) + \mathrm{j}\mathcal{X}\left(\boldsymbol{J}_n\right) = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{r}} \times \boldsymbol{E}^{\mathrm{s}}\left(\boldsymbol{J}_n\right) \quad (2)$$

and

$$\boldsymbol{E}^{\mathrm{s}}\left(\boldsymbol{J}\right) = -\frac{\mathrm{j}Z_0}{4\pi}\int_{\Omega}\left(k\boldsymbol{J} + \frac{1}{k}\nabla\nabla\cdot\boldsymbol{J}\right)\frac{\mathrm{e}^{-\mathrm{j}kR}}{R}\,\mathrm{d}\Omega. \qquad (3)$$

[5] Definition is based on fundamental property we require from CMs, *i.e.*, ability to diagonalize EFIE operator.

Short Comment on Defining GEP

Conventional approach via generalized eigenvalue problem,

$$\mathcal{X}\left(\boldsymbol{J}_n\right) = \lambda_n \mathcal{R}\left(\boldsymbol{J}_n\right) \tag{4}$$

is an alternative here, however, explicitly

$$\int_{\Omega} \left(\boldsymbol{J}_n + \frac{\nabla_{\mathrm{t}} \nabla \cdot \boldsymbol{J}_n}{k^2} \right) \frac{\cos\left(kR\right)}{R} \,\mathrm{d}\Omega = \lambda_n \int_{\Omega} \left(\boldsymbol{J}_n + \frac{\nabla_{\mathrm{t}} \nabla \cdot \boldsymbol{J}_n}{k^2} \right) \frac{\sin\left(kR\right)}{R} \,\mathrm{d}\Omega \tag{5}$$

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- Characteristic vectors are properly normalized spherical harmonics[6].

[6] M. Capek, V. Losenicky, L. Jelinek, *et al.*, "Validating the characteristic modes solvers", *IEEE Trans. Antennas Propag.*, vol. PP, no. 99, pp. 1–12, 2017. DOI: 10.1109/TAP.2017.2708094

Derived Analytical Results: Eigenvalues λ_n

Use[7] of

$$\lambda_n = \frac{\int\limits_{\Omega} \boldsymbol{J}_n^* \cdot \mathcal{X}\left(\boldsymbol{J}_n\right) \mathrm{d}\Omega}{\int\limits_{\Omega} \boldsymbol{J}_n^* \cdot \mathcal{R}\left(\boldsymbol{J}_n\right) \mathrm{d}\Omega} \tag{6}$$

[7] L. Jelinek, M. Capek, P. Hazdra, *et al.*, "An analytical evaluation of the quality factor Q_Z for dominant spherical modes", *IET Microw. Antenna P.*, vol. 9, no. 10, pp. 1096–1103, 2015. DOI: 10.1049/iet-map.2014.0302

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- for calculation of characteristic eigenvalues

$$\lambda_n^{\mathrm{TE}} = -\frac{ka\,\mathrm{y}_n\left(ka\right)}{ka\,\mathrm{j}_n\left(ka\right)}, \quad \text{and} \quad \lambda_n^{\mathrm{TM}} = -\frac{(n+1)\,\mathrm{y}_n\left(ka\right) - ka\,\mathrm{y}_{n+1}\left(ka\right)}{(n+1)\,\mathrm{j}_n\left(ka\right) - ka\,\mathrm{j}_{n+1}\left(ka\right)}, \tag{7}$$

- and characteristic eigenangles

$$\delta_n = 180 - \frac{180}{\pi}\arctan\left(\lambda_n\right). \tag{8}$$

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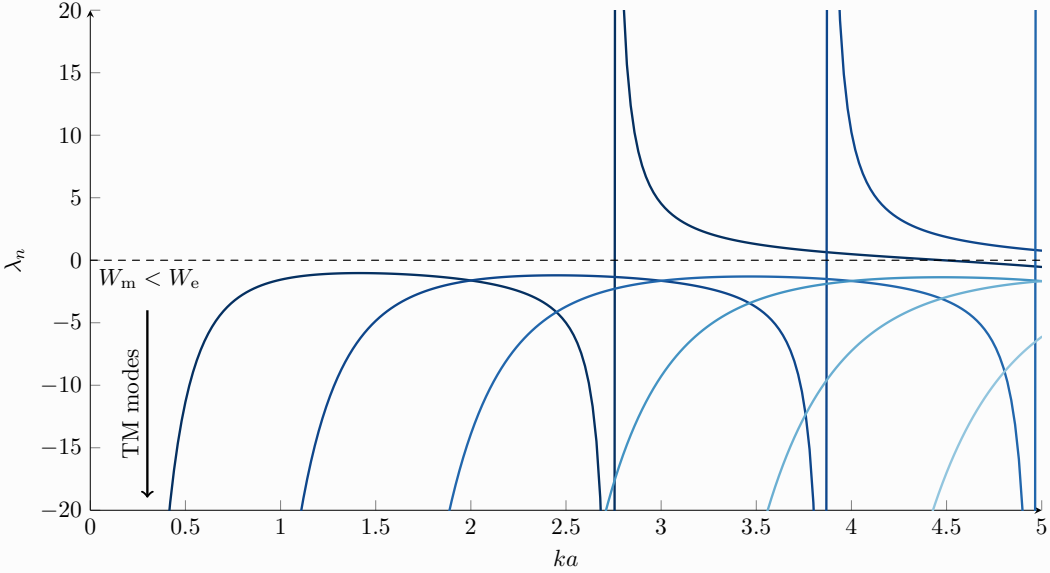
Eigenvalues λ_n



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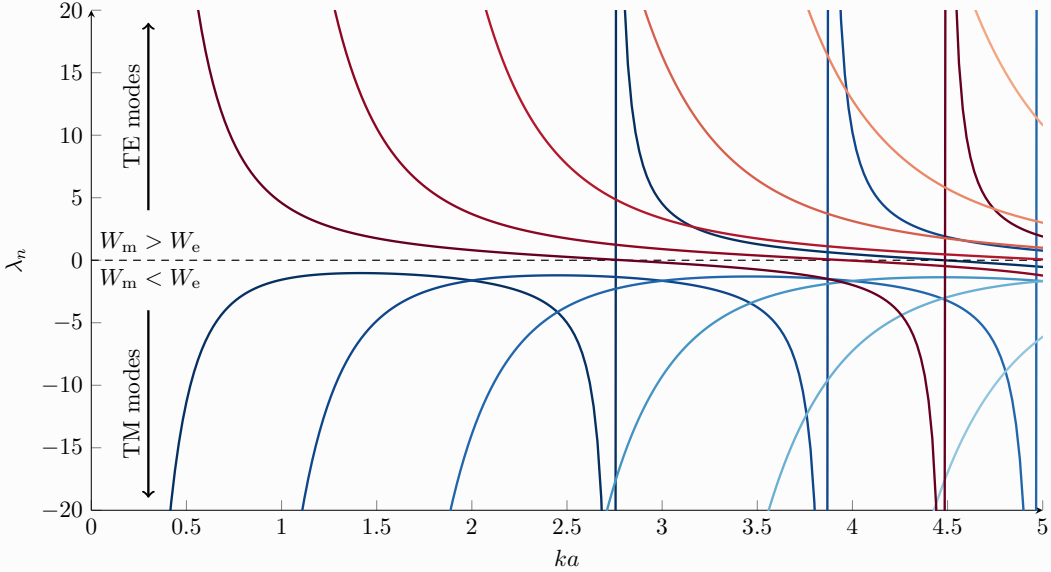
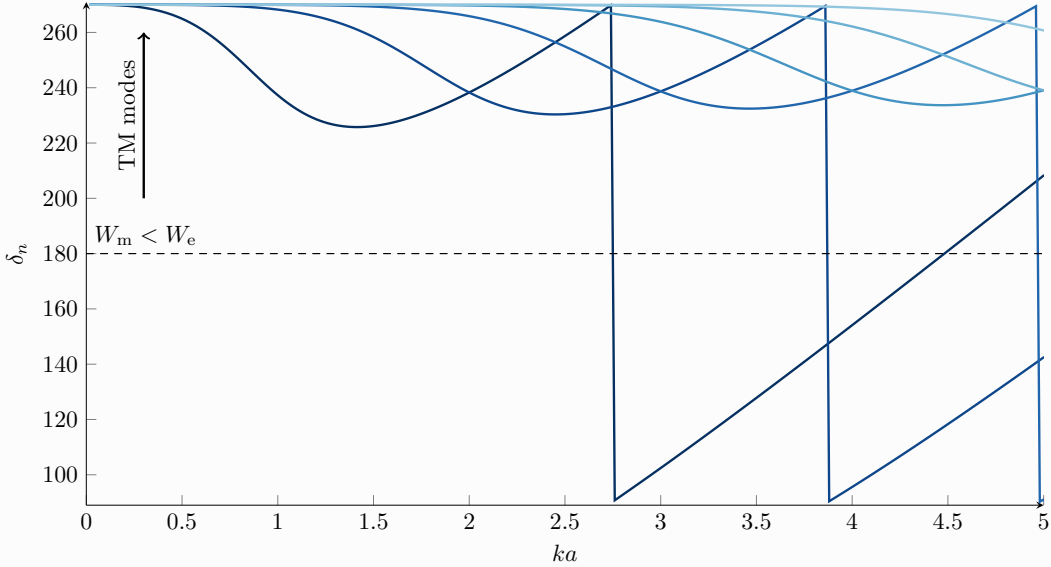


Eigenangles δ_n



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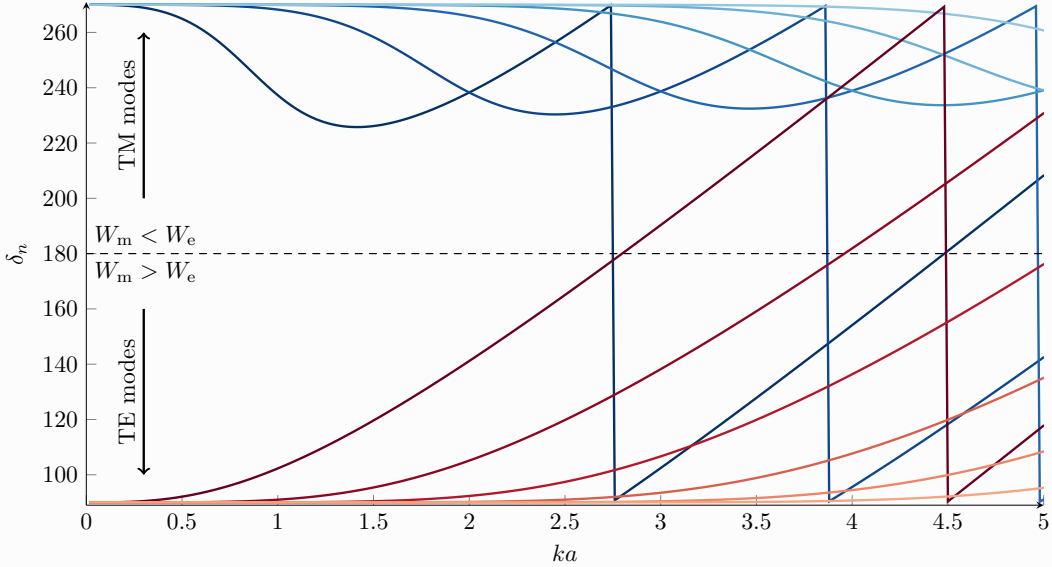


What Problems Do We Expect?

While simplest canonical body, spherical shell has plenty of potential issues, *e.g.*,

- degenerate eigenspace[8],
- conformity of spherical surface with commonly used basis functions,
- internal resonances[9],
- computationally demanding evaluation[10].

Spherical shell made of 194 triangles.

[8] K. R. Schab and J. T. Bernhard, "A group theory rule for predicting eigenvalue crossings in characteristic mode analyses", *IEEE Antennas Wireless Propag. Lett.*, no. PP, 2016, K. R. Schab, J. M. Outwater Jr., M. W. Young, *et al.*, "Eigenvalue crossing avoidance in characteristic modes", *IEEE Trans. Antennas Propag.*, vol. 64, no. 7, pp. 2617–2627, 2016. DOI: 10.1109/TAP.2016.2550098

[9] T. K. Sarkar, E. Mokole, and M. Salazar-Palma, "An expose on internal resonance, external resonance and characteristic modes", *IEEE Trans. Antennas Propag.*, vol. 64, no. 11, pp. 4695–4702, 2016. DOI: 10.1109/TAP.2016.2598281

[10] Having no junctions, spherical shell (and other closed objects) has highest possible ratio between number of basis functions (unknowns) and triangles (3/2).

What Tests[11] Do We Propose?

Test #1 Accuracy of characteristic eigenvalues λ_n,

Test #2 modal tracking of $\lambda_n(ka)$ and $\boldsymbol{J}_n(ka)$,

Test #3 conformity of characteristic currents $\boldsymbol{J}_n$

$$\chi_n = \max_p \sqrt{\sum_{q=-p}^{p} \left| \int_\Omega \hat{\boldsymbol{J}}_{pq}^{\mathrm{TM/TE}}(\boldsymbol{r}) \cdot \hat{\boldsymbol{J}}_n(\boldsymbol{r}) \, \mathrm{d}S \right|^2}, \quad \hat{\boldsymbol{J}} = \frac{\boldsymbol{J}}{\sqrt{\int_\Omega |\boldsymbol{J}(\boldsymbol{r})|^2 \, \mathrm{d}S}}, \tag{9}$$

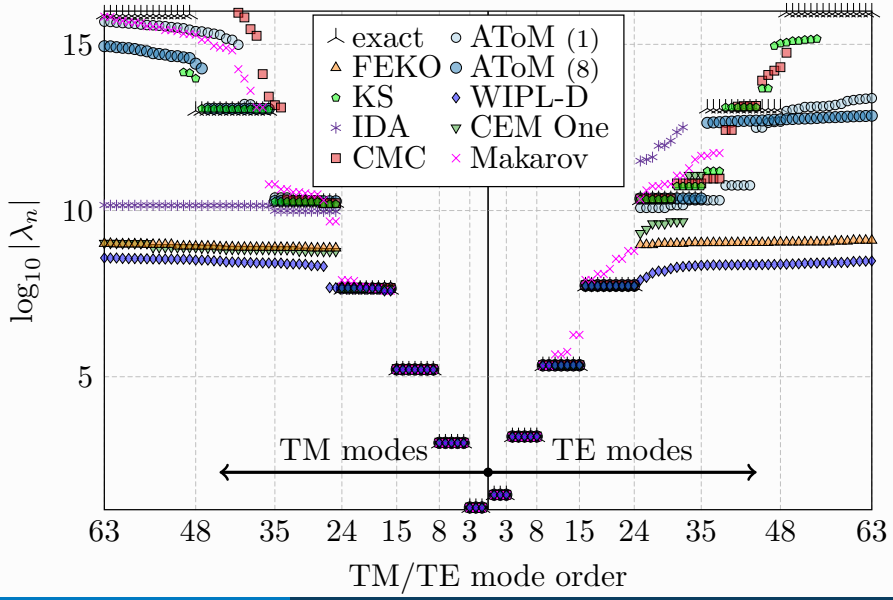
Test #4 conformity of characteristic far fields $\boldsymbol{F}_n$

$$\zeta_n = \frac{\max_p \sum_{q=-p}^{p} \left| P_{pq,n}^{\mathrm{TE/TM}} \right|^2}{\sum_{p,q} \sum_{\mathrm{TE/TM}} \left| P_{pq,n}^{\mathrm{TE/TM}} \right|^2}, \quad P_{pq,n}^{\mathrm{TE/TM}} = \frac{1}{2Z_0} \int_0^{2\pi} \int_0^{\pi} \left(\boldsymbol{F}_{pq}^{\mathrm{TE/TM}} \right)^* \cdot \boldsymbol{F}_n \sin\vartheta \, \mathrm{d}\vartheta \, \mathrm{d}\varphi. \tag{10}$$

[11] M. Capek, V. Losenicky, L. Jelinek, *et al.*, "Validating the characteristic modes solvers", *IEEE Trans. Antennas Propag.*, vol. PP, no. 99, pp. 1–12, 2017. DOI: 10.1109/TAP.2017.2708094

Test #1: Characteristic Numbers λ_n

Spherical shell, 750 basis functions (except WIPL-D), $ka = 1/2$



Test #1: Characteristic Numbers λ_n

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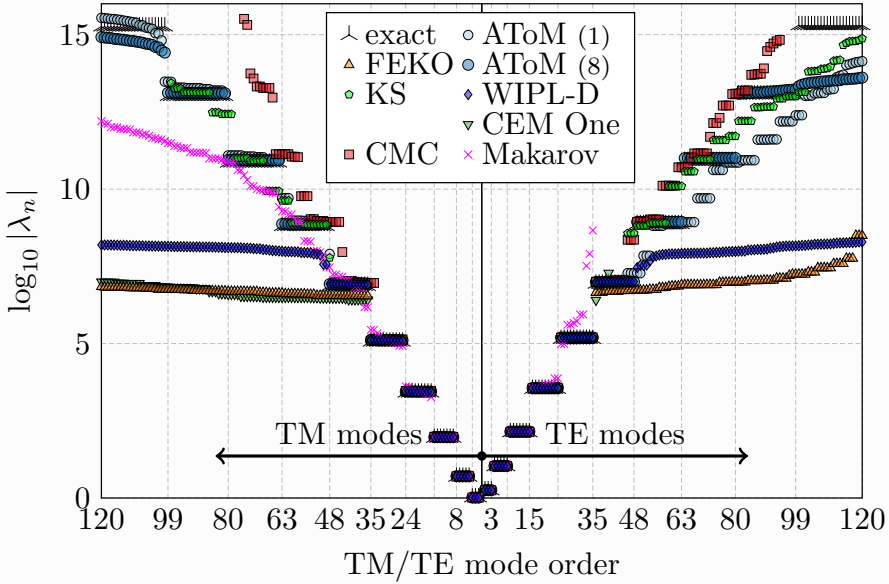
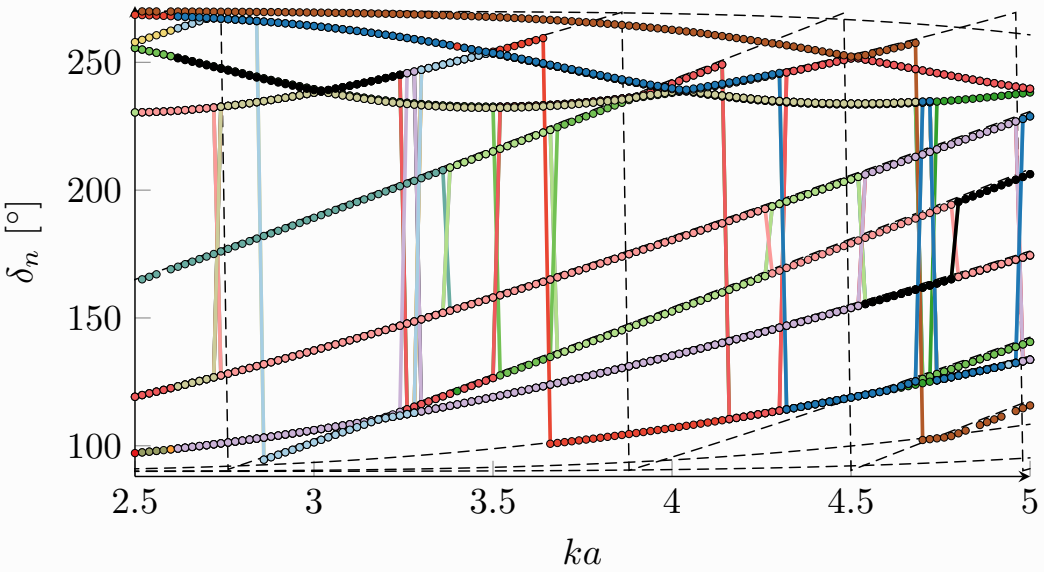


Test #2: Tracking of Eigenvalues $\lambda_n(ka)$

Spherical shell, 750 basis functions, 100 modes at 226 frequency samples from $ka = 1/2$ to $ka = 5$, FEKO



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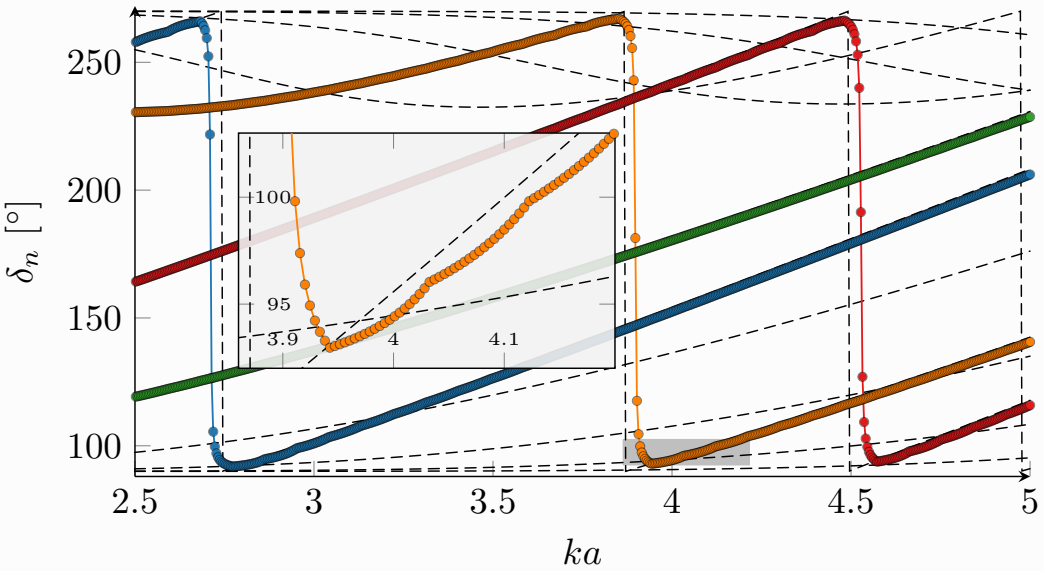

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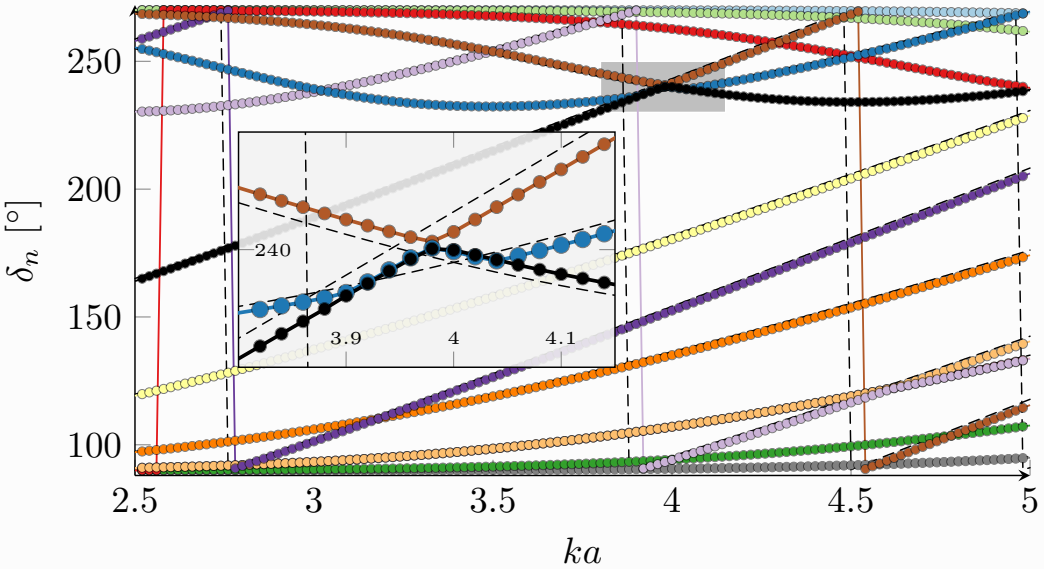
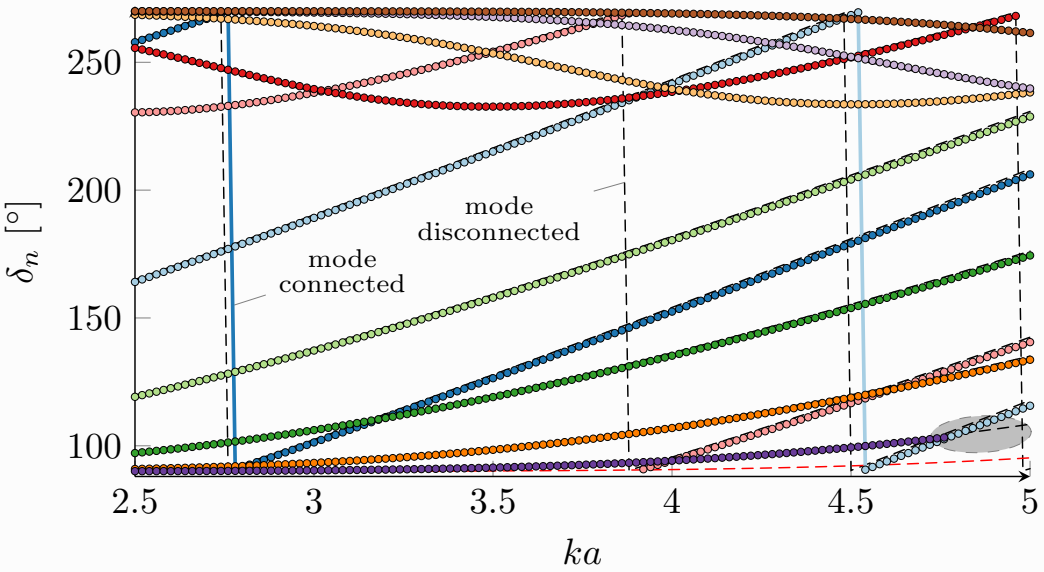
Spherical shell, higher-order basis functions, 100 modes at 226 frequency samples from $ka = 1/2$ to $ka = 5$, WIPL-D



Test #2: Tracking of Eigenvalues $\lambda_n\,(ka)$

Spherical shell, 750 basis functions, 100 modes at 226 frequency samples from $ka = 1/2$ to $ka = 5$, AToM



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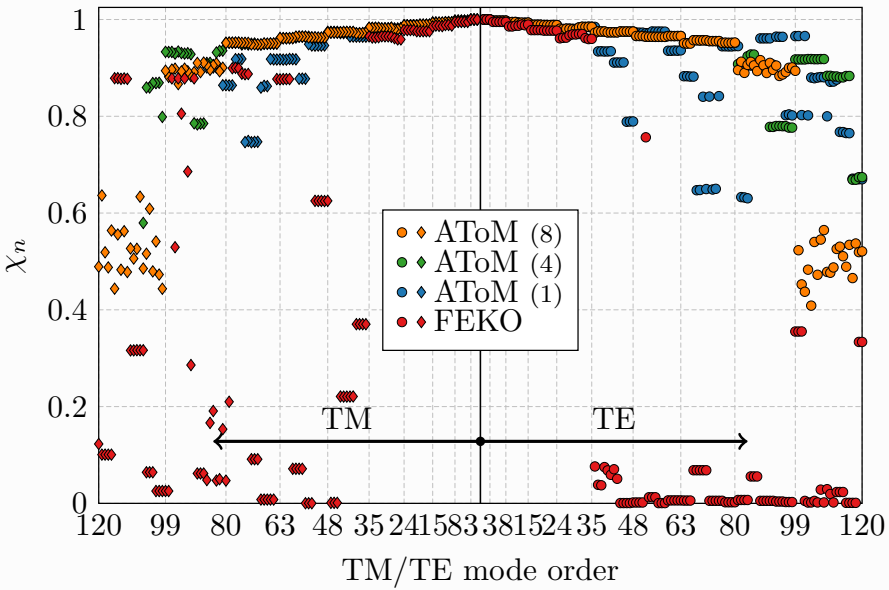

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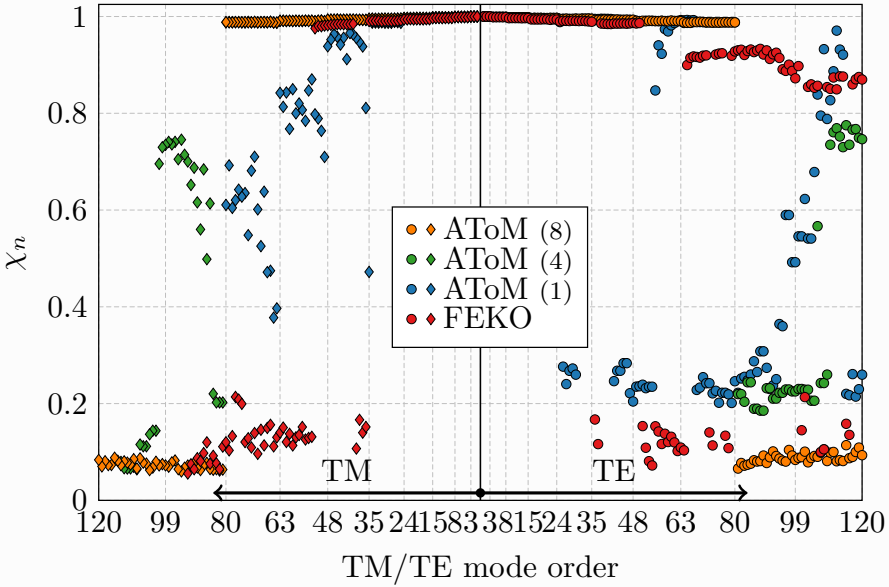

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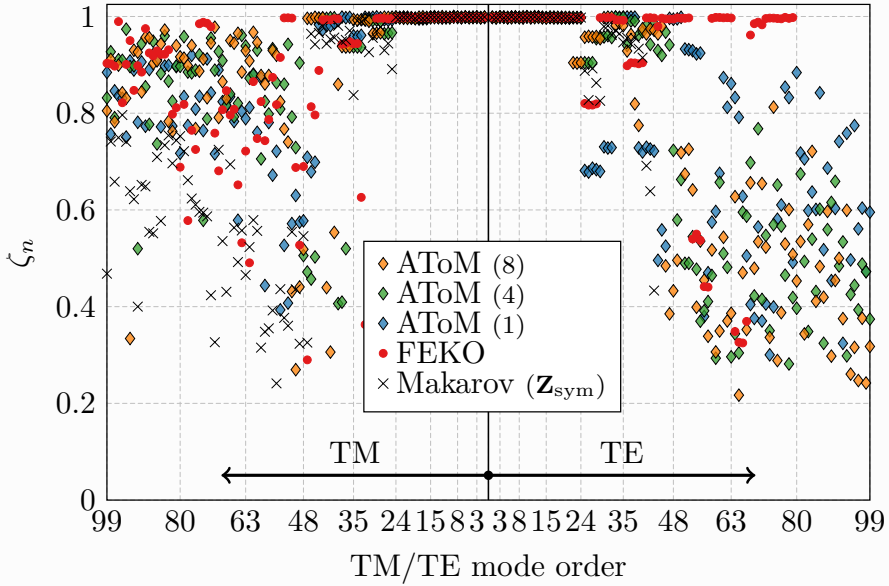

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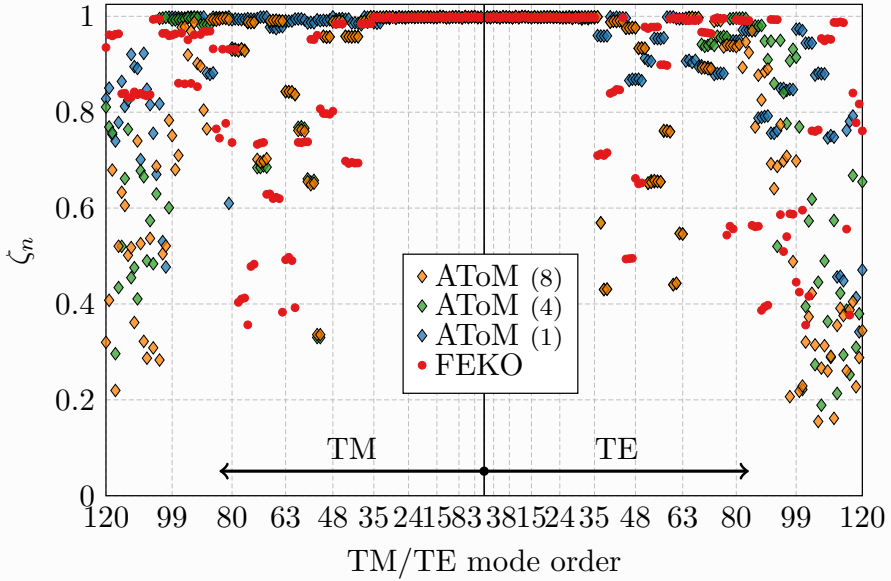

Some Observations

- The numerical dynamics of impedance matrix is essential,
- key role the used quadrature rule and (symmetric) testing,

[12] Consider the $2l + 1$ degeneracy of the TM and TE modes.

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- key role the used quadrature rule and (symmetric) testing,
- CST-MWS
  - uses CFIE and interpolation,
  - is extremely memory-consuming,
  - only 15 modes[12] have been obtained for $ka \in \{0.5, 5\}$,
  - provides no impedance matrices.
- FEKO
  - uses EFIE,
  - has surprising problems to track modes of sphere,
  - only 26 modes[12] have been obtained for $ka \in \{0.5, 5\}$.
- WIPL-D
  - delivers nice results, probably thanks to the higher-order basis functions.

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Proposed tests can be used as the *standards* for validating EM codes.

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"*How to Approach the Benchmarking Dilemma?*"

From *Benchmarking Full Wave Analysis of Periodic Structures: Non Perpendicularity at Periodic Boundaries* talk by Yahya Rahmat-Samii

1. *Select a real life example.*
2. *Create a simple and canonical version of the example.*
3. *Try to identify an analytical solution at the reference.*
4. *Compare the software results against the reference.*
5. *Identify potential shortcomings.*
6. *Bring this to the attention of community and software vendors.*
7. *Make this as an example to be archived for others to test their codes.*
8. *Be clear that this is your best effort and take no responsibility.*

- Notice there are IEEE standards[13] for validation of CEM software.

[13] *1597.1-2008 – IEEE standard for validation of computational electromagnetics computer modeling and simulations*, IEEE, 2008. DOI: 10.1109/IEESTD.2008.4957854
1597.2-2010 – IEEE recommended practice for validation of computational electromagnetics computer modeling and simulations, IEEE, 2011. DOI: 10.1109/IEEESTD.2011.5721917

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From *Benchmarking Full Wave Analysis of Periodic Structures: Non Perpendicularity at Periodic Boundaries* talk by Yahya Rahmat-Samii

1. *Select a real life example.*
   - characteristic modes decomposition
2. *Create a simple and canonical version of the example.*
   - characteristic mode decomposition of a spherical shell
3. *Try to identify an analytical solution at the reference.*
   - done via spherical harmonics and radiation power normalization
4. *Compare the software results against the reference.*
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   - steps 4.–6. done[14,15]
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   - done[15]
8. *Be clear that this is your best effort and take no responsibility.*

[14] M. Capek, V. Losenicky, L. Jelinek, *et al.*, "Validating the characteristic modes solvers", *IEEE Trans. Antennas Propag.*, vol. PP, no. 99, pp. 1–12, 2017. DOI: 10.1109/TAP.2017.2708094

[15] *Materials for characteristic modes benchmarking*, 2017. [Online]. Available: https://elmag.fel.cvut.cz/CMbenchmark

Conclusion

Spherical shell makes it possible to test our codes against analytical data:

- meshing schemes, basis functions, singularities, integration rules[16],
- properties of CM decomposition, tracking routines[16],
- hardware (CPU, RAM, HDD) requirements,
- different commercial and in-house packages, ▸ Webpage with benchmarks & results
- No feeding mechanism needed.

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Ongoing work

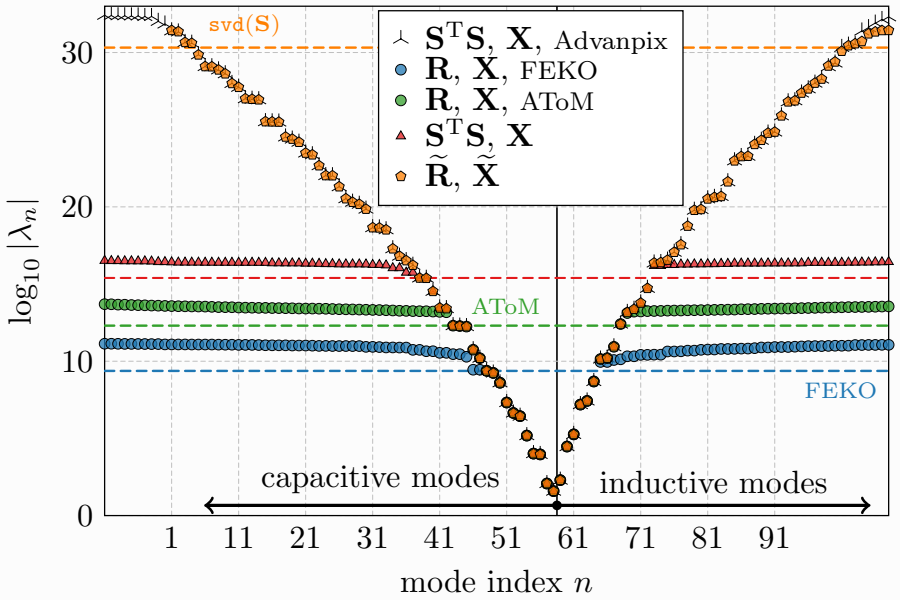
- construction of projection matrix with favorable numerical properties,
- complete monitoring of all systematical errors in AToM[17],
- think about solution to conical system.

[16] M. Capek, V. Losenicky, L. Jelinek, *et al.*, "Validating the characteristic modes solvers", *IEEE Trans. Antennas Propag.*, vol. PP, no. 99, pp. 1–12, 2017. DOI: 10.1109/TAP.2017.2708094

[17] (2017). Antenna Toolbox for MATLAB (AToM), , Czech Technical University in Prague, [Online]. Available: www.antennatoolbox.com

What We Are Working on Now...

Characteristic numbers λ_n of rectangular plate $L/W = 2$, $ka = 0.5$, 655 basis functions



Questions?

For complete PDF presentation see capek.elmag.org

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Systematic Shift of Eigenvalues – Errors in Model