Current Optimization for Electrically Small Antennas

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Outline

1. Current Optimization
2. Minimum Quality Factor $Q$
3. Modal Approach
4. Optimal Composition of Modes
5. On the Natural Bases
6. Summary and Concluding Remarks

In this talk:
- electric currents in vacuum,
- only surface regions are treated,
- time-harmonic quantities, \( \mathbf{A}(r, t) = \text{Re} \{ \mathbf{A}(r) \exp(j\omega t) \} \) are considered,
- be extremely careful when comparing different sources (papers): different notation!
“Is it beneficial to be here today?”

Do you know the following publications well?

Antenna Analysis × Synthesis and Antenna Design

Antenna analysis × antenna synthesis.
**Antenna Analysis × Synthesis and Antenna Design**

Antenna analysis studies...

- antenna parameters for a given antenna.

Antenna analysis × antenna synthesis.
Antenna Analysis $\times$ Synthesis and Antenna Design

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Antenna synthesis seeks for...
- optimal currents,
- optimal feeding (placement),
- optimal material.

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Antenna Analysis × Synthesis and Antenna Design

Antenna analysis studies...
- antenna parameters for a given antenna.

Antenna synthesis seeks for...
- optimal currents,
- optimal feeding (placement),
- optimal material.

Antenna design tries to find...
- the optimal combination of shape, material and feeding from infinitely many candidates.
Historical Overview

Former approaches to antenna design predominantly made use of

- circuit quantities\(^1\) \((V_{\text{in}}, Z_{\text{in}}, \Gamma, \ldots)\) → equivalent circuits,

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Former approaches to antenna design predominantly made use of

- circuit quantities\(^1\) \((V_{\text{in}}, Z_{\text{in}}, \Gamma, \ldots) \rightarrow \) equivalent circuits,
- field quantities\(^2\) \((\mathbf{E}, \mathbf{H})\).

However,

- all antenna parameters can be inferred from source current\(^3\) \((\mathbf{J}, \mathbf{M})\)

\[
p = \langle \mathbf{J}, \mathbf{L}(\mathbf{J}) \rangle.
\]

\[
\langle \mathbf{f}, \mathbf{L}(\mathbf{g}) \rangle = \int_{\Omega} \mathbf{f}^* (\mathbf{r}) \cdot \mathbf{L}(\mathbf{g}(\mathbf{r})) \, dV
\]


All antenna parameters can be inferred directly from source current

\[ p = \langle J, \mathcal{L}(J) \rangle. \]  

(1)
Operators to Rule Them All . . .

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Observations:
- only properties of the operators are important,
- physics is imprinted in their structure,
- can be represented in many different ways, e.g., \[ \langle \psi_m, \mathcal{L} \psi_n \rangle, \] \[ \langle J_p, \mathcal{L} J_q \rangle, \]
- as compared to fields, the current support is limited.

\[ \langle f, g \rangle = \int_{\Omega} f^* (r) \cdot g (r) \, dV \]
Example: Radiated and Reactive Power

Consider Electric Field Integral Equation\(^4\) written as

\[
\mathcal{Z}(\mathbf{J}) = -\hat{n} \times \hat{n} \times \mathbf{E}^i(\mathbf{J})
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and let us represent it in a basis \(\{\psi_n\}, n \in \{1, \ldots, N\}\) as \(\mathbf{Z} = [Z_{mn}]\), in which

\[
Z_{mn} = \langle \psi_m, \mathcal{Z}(\psi_n) \rangle = -\frac{jZ_0}{4\pi} \int_{\Omega} \int_{\Omega'} \left( k \psi_m^* \cdot \psi_n - \frac{1}{k} \nabla \cdot \psi_m^* \nabla' \cdot \psi_n \right) \frac{e^{-jk|\mathbf{r} - \mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \ d\Omega' \ d\Omega
\]

(3)

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\]

or even more as

\[
(1 + j\lambda_m) \delta_{mn} = \frac{1}{2} \langle \mathbf{I}_m, \mathbf{ZI}_n \rangle = \frac{1}{2} \mathbf{I}_m^H \mathbf{ZI}_n.
\]

Example: Radiated and Reactive Power

Consider Electric Field Integral Equation\textsuperscript{4} written as

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and let us represent it in a basis \{\psi_n\}, \( n \in \{1, \ldots, N\} \) as \( \mathbf{Z} = [Z_{mn}] \), in which

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\begin{itemize}
  \item All common algebraic operations are available for (3).
  \item Representation (4) profitably diagonalizes impedance operator (matrix).
\end{itemize}

Current Optimization for Electrically Small Antennas

Source Concept $(\mathcal{I}, \mathcal{M})$

- Perspective topology and geometry
- Modal decompositions
- HPC, algorithm efficiency
- Heuristic or convex optimization
- Integral and variational methods
Source Concept

Sketch of main fields of the source concept.

Source concept

\( \mathbf{J} \) completely by a source currents \( \mathbf{J} \) (and \( \mathbf{M} \)),

\[ \mathbf{J} \equiv \mathbf{M} \equiv 0 \iff r \notin \Omega. \]
A current $J = J(r, \omega)$, $r \in \Omega$, is denoted $J_{\text{opt}}$ and called as optimal current iff

$$\langle J_{\text{opt}}, \mathcal{L}(J_{\text{opt}}) \rangle = \min_J \langle J, \{\mathcal{L}(J)\} \rangle = p_{\text{min}},$$  \hfill (5)

$$\langle J_{\text{opt}}, \mathcal{M}_n(J_{\text{opt}}) \rangle = q_n,$$  \hfill (6)

$$\langle J_{\text{opt}}, \mathcal{N}_n(J_{\text{opt}}) \rangle \leq r_n.$$  \hfill (7)
Optimal Currents – What They Are?

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- What are the optimal currents good for?
  - They establish fundamental bounds of $p = \mathcal{L}(\mathbf{J})$ for a given $\Omega$, $\omega$.

- How to find them?
  - You will see...

- Can they be realized?
  - Only as impressed currents (they are unrealistic).
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► How to find them?
  - You will see…

► Can they be realized?
  - Only as impressed currents (they are unrealistic).

The rest of the presentation is about $\mathcal{L}$ and techniques how to find $J_{opt}$. 

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Case Study: Minimization of Quality Factor $Q$

Quality factor $Q$
- is (generally) inversely proportional to fractional bandwidth (FBW),
- therefore, of interest for electrically small antennas (ESA, $ka < 1/2$).
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- therefore, of interest for electrically small antennas (ESA, $ka < 1/2$).

Current $J_{\text{opt}}$ minimizing quality factor $Q$ of a given shape $\Omega$:

$$Q(J_{\text{opt}}) = \min_{J} \{Q(J)\} \quad (8)$$

How to find $J_{\text{opt}}$ for a given $\Omega$?
Procedure Undertaken in This Presentation

Procedure followed in this talk:

**STEP 1** representation of operators as matrices,
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STEP 4 representation of a solution in an appropriate basis,
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**STEP 5** optimal composition of modal currents,
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STEP 1 representation of operators as matrices,
STEP 2 definition of quality factor $Q$ and stored energy $W_{sto}$,
STEP 3 formulation of optimization task related to (8),

STEP 4 representation of a solution in an appropriate basis,
STEP 5 optimal composition of modal currents,
  1st method: combination of 2 modes\(^5\),
  2nd method: composition of $n$ modes\(^6\),
  3rd method: solution with 1 mode\(^7\).


Source Concept Based Definition – Operators

STEP 1

► assume operators in their matrix forms, i.e., $Z \rightarrow Z$,
► assume functions in their vector forms, i.e., $J \rightarrow I$,

\[
Z = R + jX = R + j(X_m - X_e), \quad J \approx \sum_n I_n \psi_n,
\]  

therefore, we get

\[
P_r = \frac{1}{2} I^H R I, \quad 2\omega (W_m - W_e) = \frac{1}{2} I^H (X_m - X_e) I
\]  

(9) (10)
**Source Concept Based Definition – Operators**

**STEP 1**
- assume operators in their matrix forms, \( \mathcal{Z} \rightarrow \mathcal{Z} \),
- assume functions in their vector forms, \( \mathcal{J} \rightarrow \mathbf{I} \),

\[ \mathcal{Z} = \mathbf{R} + j\mathbf{X} = \mathbf{R} + j(\mathcal{X}_m - \mathcal{X}_e), \quad \mathcal{J} \approx \sum_n I_n \psi_n, \quad (9) \]

therefore, we get

\[ P_r = \frac{1}{2} \mathbf{I}^H \mathbf{R} \mathbf{I}, \quad 2\omega (W_m - W_e) = \frac{1}{2} \mathbf{I}^H (\mathcal{X}_m - \mathcal{X}_e) \mathbf{I} \quad (10) \]

**STEP 2A**
- definition of quality factor \( Q \)

\[ Q(\mathbf{I}) = \frac{2\omega \max\{W_m, W_e\}}{P_r} = \frac{\omega (W_m + W_e)}{P_r} + \frac{\omega |W_m - W_e|}{P_r} \quad (11) \]
STEP 2B

- frankly speaking we still do not have complete idea what the stored energy is

\[
W_m + W_e = W_{stot} \approx \frac{1}{4} \left\langle J, \partial \text{Im} \{Z\} \partial \omega \right\rangle \approx \frac{1}{4} I_H \partial X \partial \omega = \frac{1}{4} I_H X' I(12)
\]

STEP 3

- find \( I_{opt} \) so that

\[
\text{minimize quality factor } Q(I)
\]

subject to

\[
W_m(I) - W_e(I) = 0
\]

(14)
Stored Energy Operator and Its Minimization

STEP 2B

- frankly speaking we still do not have complete idea what the stored energy is
  - but… we have many papers attempting to define it\(^8\)
  - one of the best possibilities for small radiators, \(ka < 1\), is\(^9\):

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\]

\((12)\)

---


Minimum Quality Factor $Q$

Stored Energy Operator and Its Minimization

STEP 2B

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STEP 3

- find $\mathbf{I}_{\text{opt}}$ so that

$$\text{minimize quality factor } Q(\mathbf{I}),$$

subject to

$$W_m(\mathbf{I}) - W_e(\mathbf{I}) = 0.$$  \hspace{1cm} (14)

---


Modal Approach: Combining Two Proper Modes

STEP 4

- let us decompose the current into (yet unknown) modes such that

\[ I = \sum_{n=1}^{N} \alpha_n I_n \]  \hfill (15)

then, substituting (10), (12), and (15) into (11), the quality factor \( Q \) reads

\[ Q(I) = \frac{\omega V \sum_{v=1}^{V} U \sum_{u=1}^{U} \alpha^* u \alpha v I_H u X' I_v}{\sqrt{\left( \sum_{v=1}^{V} U \sum_{u=1}^{U} \alpha^* u \alpha v I_H u X I_v \right)^2}} \]  \hfill (16)

analytical solution can easily be found as a combination of two modes iff

\[ I_H u R I_v = \delta_{uv}, \quad I_H u X I_v = \Lambda_{uv} \delta_{uv}, \quad \omega I_H u X' I_v = \chi_{uv} \delta_{uv}. \]  \hfill (17)
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Modal Approach: Combining Two Proper Modes

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Why to Combine Two Modes?

Tuning by external lumped element (localized current).
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Tuning by external lumped element (localized current).

Tuning by distributive current.

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Current Optimization for Electrically Small Antennas
How to Combine Two Modes?

Normalizing $\alpha_1 = 1$ and selecting proper mode(s), we get\(^{10}\)

$$Q(I_{\text{opt}}) = \frac{\omega \left( I_1^H X' I_1 + |\alpha_{\text{opt}}|^2 I_2^H X' I_2 \right)}{2 \left( 1 + |\alpha_{\text{opt}}|^2 \right)}.$$  \hspace{1cm} (18)

How to Combine Two Modes?

Normalizing $\alpha_1 = 1$ and selecting proper mode(s), we get\(^\text{10}\)

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**STEP 5**

To diagonalize at least two of $R$, $X$, and $X'$ matrices we can choose:

$$XI_u = \lambda_u R I_u,$$  (19)

$$\frac{\omega}{2} X' I_u = q_u R I_u,$$  (20)

$$XI_u = \xi_u \frac{\omega}{2} X' I_u.$$  (21)

---

Modal Approach

How to Combine Two Modes?

Normalizing $\alpha_1 = 1$ and selecting proper mode(s), we get

$$Q(I_{opt}) = \frac{\omega \left( I_1^H X' I_1 + |\alpha_{opt}|^2 I_2^H X' I_2 \right)}{2 \left( 1 + |\alpha_{opt}|^2 \right)}.$$  \hspace{1cm} (18)

STEP 5

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$$XI_u = \xi_u \frac{\omega}{2} X' I_u.$$ \hspace{1cm} (21)

---

Example: Optimal Current For Minimal Quality Factor $Q$

Take the pen and try to draw a current possessing minimum quality factor $Q$...

PEC plate $L \times L/2$, $ka = 0.5$. 
Example: Optimal Current For Minimal Quality Factor $Q$

...here is the correct answer.

Optimal current with respect to minimum quality factor $Q$. 
Modal Approach

Modal Composition of the Optimal Current

Dominant (dipole-like) characteristic mode $J_1, \alpha_1 = 1$.

First inductive (loop-like) mode $J_2, \alpha_2 = 0.4553$. 
Modal Approach

Modal Composition of the Optimal Current

- Dominant (dipole-like) characteristic mode \( J_1, \alpha_1 = 1 \).
- First inductive (loop-like) mode \( J_2, \alpha_2 = 0.4553 \).

- Characteristic modes (19) are quite convenient to define and interpret optimal currents\(^\text{11}\), e.g., for \( ka < 1 \) we have \( Q(J_{\text{opt}}) \approx Q(J_1 + \alpha_2 J_2) \)
- However, for higher \( ka \) or for highly irregular shapes, the energy cross-terms occur, i.e., \( \omega I_u^H X^I_I \neq 0 \)

Convexity of Two-Mode Combination

Quality factor $Q$ as a combination of two modes for $L \times L/2$ PEC plate, $Q_U(I) = \omega I^H X' I / 2 I^H R I$, $I = I_1 + \alpha_2 I_2$. 

$k a = 0.5$

$I_{C1} + \alpha_2 I_{I1}$

$I_{C2} + \alpha_2 I_{I1}$

$I_{C1} + \alpha_2 I_{C2}$

$I_{C1} + \alpha_2 I_{I2}$

$I_{C2}$

$I_{I1}$

$I_{I1}$

$I_{I2}$

$I_{C1} + \alpha_2 I_{I2}$

$\alpha_{opt}$

$Q_U(I)$

$Q(I)$

40

54.73

24.42

35.60

43.89

3.5 0.5 0.0 5 1

80

$I_{C1}$

$I_{C2}$

$I_{I1}$

$I_{I2}$

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- extremely straightforward analytical solution for spherical shell\textsuperscript{12}
- sub-optimality for $G/Q$ ratio
- alternative bases can be used to reduce the effect of cross-terms
- intuitive, however, non-convex and only approximative procedure (non-zero cross-terms)

Combining More Modes

Lessons Learned

- More modes than two are needed for a given set of operators.
- Particular approach to quality factor $Q$ needs to be generalized.
Lessons Learned

- More modes than two are needed for a given set of operators.
- Particular approach to quality factor $Q$ needs to be generalized.

For now, let us start with the following general optimization problem\textsuperscript{13}:

$$\min_{\mathbf{I}} \{ \mathbf{I}^H \mathbf{A} \mathbf{I} \},$$

$$\mathbf{I}^H \mathbf{B} \mathbf{I} = 1,$$

$$\mathbf{I}^H \mathbf{C} \mathbf{I} = \gamma.$$

- any problem expressible in bilinear form can be solved,
- matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ cannot be generally diagonalized simultaneously.

represent our solution in a basis \( (\text{same as before}) \)

\[ I = \sum_{n=1}^{N} \alpha_n I_n \]  \hspace{1cm} (25)
represent our solution in a basis (same as before)

\[ I = \sum_{n=1}^{N} \alpha_n I_n \]  

so that the modal currents fulfill (before: \( S \equiv X/2, \ T \equiv R/2 \))

\[ SI_n = \zeta_n TI, \quad I_m^H (T + jS) I_n = (1 + j\zeta_n) \delta_{mn} \]  

with \( T = T^H, \ S = S^H \) and \( \zeta_n \in \mathbb{R} \)
Optimal Composition of Modes

Optimization Problem to Be Solved

- represent our solution in a basis (same as before)
  \begin{equation}
  \mathbf{I} = \sum_{n=1}^{N} \alpha_n \mathbf{I}_n \tag{25}
  \end{equation}

- so that the modal currents fulfill (before: \( S \equiv \mathbf{X}/2, \mathbf{T} \equiv \mathbf{R}/2 \))
  \begin{align*}
  \mathbf{S}\mathbf{I}_n &= \zeta_n \mathbf{T}\mathbf{I}_n, \\
  \mathbf{I}_m^H (\mathbf{T} + j\mathbf{S}) \mathbf{I}_n &= (1 + j\zeta_n) \delta_{mn} \tag{26}
  \end{align*}

with \( \mathbf{T} = \mathbf{T}^H, \mathbf{S} = \mathbf{S}^H \) and \( \zeta_n \in \mathbb{R} \)

- finally, represent operators \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \) in basis (25) as
  \begin{align*}
  \mathbf{A}_{\text{GEP}}^{mn} &= \langle \mathbf{I}_m, \mathbf{A}\mathbf{I}_n \rangle = \mathbf{I}_m^H \mathbf{A}\mathbf{I}_n, \\
  \mathbf{A}^{\text{GEP}} &= \begin{bmatrix} \mathbf{A}_{\text{GEP}}^{mn} \end{bmatrix} \tag{27}
  \end{align*}

and same for \( \mathbf{B} \rightarrow \mathbf{B}^{\text{GEP}} \) and \( \mathbf{C} \rightarrow \mathbf{C}^{\text{GEP}} \)
Problem (22)–(24) is transformed to

\[ \min_{\alpha} \{ \alpha^H A^{\text{GEP}} \alpha \}, \quad (28) \]

\[ \alpha^H B^{\text{GEP}} \alpha = 1, \quad (29) \]

\[ \alpha^H C^{\text{GEP}} \alpha = \gamma, \quad (30) \]

\footnote{Be careful here with \( \lambda_1, \lambda_2 \): here, they present the Lagrange multipliers not the characteristic numbers as in the previous method.}
Optimization Procedure

Problem (22)–(24) is transformed to and solved using Lagrange multipliers

$$\min_\alpha \{ \alpha^H A^{\text{GEP}} \alpha \}, \quad (28) \quad A^{\text{GEP}} \alpha = \lambda_1 C^{\text{GEP}} \alpha + \lambda_2 B^{\text{GEP}} \alpha. \quad (31)$$

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\alpha^H B^{\text{GEP}} \alpha = 1, \quad (29)
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\[
\alpha^H C^{\text{GEP}} \alpha = \gamma, \quad (30)
\]

Procedure proceed in following steps:

1. Choose \( \lambda_2 \) and solve (31).
2. Normalize all solutions to satisfy (29).
3. Check the constraint (30).
4. Vary \( \lambda_2 \) and find solution to (30).
5. From candidates satisfying (29)–(31) select the one fulfilling (28).

---

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Minimum Quality Factor $Q$ and Lagrange Multipliers

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<td>$A = \omega X'/4$,  (32)</td>
<td>$S = X/2$, (35)</td>
<td>$\gamma = 0$, (37)</td>
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<td>$B = R/2$,          (33)</td>
<td>$T = R/2$, (36)</td>
<td>$B^{\text{GEP}} = I$, (38)</td>
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<td>$C = X/2$,          (34)</td>
<td></td>
<td>$\lambda_2 = \alpha^H A^{\text{GEP}} \alpha = Q$, (39)</td>
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A^{\text{GEP}} \alpha = \lambda_1 C^{\text{GEP}} \alpha + \lambda_2 B^{\text{GEP}} \alpha
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## Minimum Quality Factor $Q$ and Lagrange Multipliers

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### Observations:

- $\gamma = 0$ is the resonant condition,
- With $\gamma = 0$ the multiplier $\lambda_2$ is directly equal to the optimized quantity.

---

$$A^{\text{GEP}} \alpha = \lambda_1 C^{\text{GEP}} \alpha + \lambda_2 B^{\text{GEP}} \alpha$$

$$\alpha^H A^{\text{GEP}} \alpha = \lambda_1 \alpha^H C^{\text{GEP}} \alpha + \lambda_2 \alpha^H B^{\text{GEP}} \alpha$$
Optimal Composition of Modes

Procedure and Its Results

Optimization done as $\lambda_2(\lambda_1) = Q$. 

$\lambda_2 = Q$
$ka = 0.5$

1. vary $\lambda_1$, find $\lambda_2$, i.e., tuning for minimum quality factor $Q$

2. vary $\lambda_2$, find $\lambda_1$, i.e., sweeping $\lambda_2 = Q$ for proper tuning

Capek, M., CTU in Prague

Current Optimization for Electrically Small Antennas
Procedure and Its Results

- procedure converges well to the correct result
- two options how to solve the problem
  1. vary $\lambda_1$, find $\lambda_2$, i.e., tuning for minimum quality factor $Q$
  2. vary $\lambda_2$, find $\lambda_1$, i.e., sweeping $\lambda_2 = Q$ for proper tuning

Optimization done as $\lambda_2 (\lambda_1) = Q$. 
Lessons Learned

▶ Lagrange multipliers constitute powerful method.
▶ It is, however, both theoretically and practically quite complicated.
▶ Is there any basis in which $I_{opt} = I_1$?
What Is the Best Basis Possible?

Lessons Learned

▶ Lagrange multipliers constitute powerful method.
▶ It is, however, both theoretically and practically quite complicated.
▶ Is there any basis in which \( I_{\text{opt}} = I_1 \)?

Procedure not relying on optimal composition of modes exists\(^{16}\)

\[
(1 - \nu) X_m + \nu X_e \, I_n = \tilde{Q}_\nu RI_n, \tag{40}
\]

constituting a dual problem which can be easily solved (since its convex nature):

\[
\min_{\{Q\}} = \max_\nu \left\{ \tilde{Q}_\nu \right\}. \tag{41}
\]

---

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constituting a dual problem which can be easily solved (since its convex nature):

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\min_I \{Q\} = \max_\nu \{\tilde{Q}_\nu\}.
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- based only on convex combination of $X_m$ and $X_e$ operators
- needs positive-definite operator in RHS of (40)

\(^{16}\text{M. Gustafsson, D. Tayli, C. Ehrenborg, et al., “Antenna current optimization using MATLAB and CVX”, , } FERMAT,\text{ vol. 15, no. 5, pp. 1–29, 2016}
Does It Work?

Originally concluded\(^{17}\) that a non-zero dual gap exists . . .

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\[ Q_{m/e} = \frac{I^H_x x_{m/e} I^H}{I^H_R I_L} \]

\[ kL \approx 0.628 \]

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\[ L/2 \]

\[ L/4 \]

\[ L \]

\[ L/2 \]

\[ \nu = 0.75 \]

\[ \nu = 0.95 \]

\[ \nu \]

\[ kL \approx 0.628 \]

\[ \min \{Q\} \]

\[ \max \{Q_{m/v}, Q_{e/v}\} \]

\[ \tilde{Q}_v \]

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\[ \nu \]

\[ kL \approx 0.628 \]

\[ \nu \]

…or not\textsuperscript{18}?


Does It Work?

Originally concluded\textsuperscript{17} that a non-zero dual gap exists…

\begin{itemize}
  \item Do you have any idea where is the problem?
\end{itemize}

\textsuperscript{17}M. Gustafsson, D. Tayli, C. Ehrenborg, \textit{et al.}, “Antenna current optimization using MATLAB and CVX”, \textit{FERMAT}, vol. 15, no. 5, pp. 1–29, 2016

Degenerated Eigenspace: Let’s Make It Work . . .

- if dual gap exists then only because of internal symmetries and existence of degenerated eigenspace\(^\text{19}\)
  - the internal symmetries must be preserved by the used discretization scheme (e.g., rectangular plate is discretized with rooftop basis functions)

Degenerated Eigenspace: Let’s Make It Work...

- if dual gap exists then only because of internal symmetries and existence of degenerated eigenspace\(^{19}\)
  - the internal symmetries must be preserved by the used discretization scheme (e.g., rectangular plate is discretized with rooftop basis functions)

- it can be shown the that dual gap can always be done zero
  - for two degenerated solutions \(\mathbf{I}_1\) and \(\mathbf{I}_2\) find \(c_2 \in \mathbb{C}\) such that\(^{20}\)

\[
\mathbf{I}_\nu^H (\mathbf{X}_m - \mathbf{X}_e) \mathbf{I}_\nu = 0, \quad \mathbf{I}_\nu = \mathbf{I}_1 + c_2 \mathbf{I}_2,
\]

\(^{20}\) Notice that \(\mathbf{I}_1\) and \(\mathbf{I}_2\) in (42) are different from those in (18).


\(^{19}\) Notice that \(\mathbf{I}_1\) and \(\mathbf{I}_2\) in (42) are different from those in (18).
**What Can Be Solved Now?**

<table>
<thead>
<tr>
<th>RHS of (40)</th>
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<tbody>
<tr>
<td>Radiated power</td>
<td>$R$</td>
</tr>
<tr>
<td>Mode expansion</td>
<td>$\frac{1}{Z_0} \sum_\tau F^H \tau F_{\tau}$</td>
</tr>
<tr>
<td>Radiation intensity</td>
<td>$U(\hat{r}, \hat{e}) = \frac{1}{Z_0} F^H (\hat{r}, \hat{e}) F (\hat{r}, \hat{e})$</td>
</tr>
<tr>
<td>Antenna gain</td>
<td>$4\pi \left( U(\hat{r}, \hat{\theta}) + U(\hat{r}, \hat{\phi}) \right)$</td>
</tr>
<tr>
<td>Ohmic losses</td>
<td>$R_\sigma$</td>
</tr>
<tr>
<td>Field intensity</td>
<td>$N^H_{e/m} N_{e/m}$</td>
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As far as the operator on the RHS of (40) is positive definite the procedure works well.
Recall the method of Lagrange multipliers from (22)–(24) as

\[ AI = \lambda_1 CI + \lambda_2 BI. \]  

(43)
Recall the method of Lagrange multipliers from (22)–(24) as

\[ \text{AI} = \lambda_1 \text{CI} + \lambda_2 \text{BI}. \]  
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Minimization of quality factor \( Q \) (proportional to \( \lambda_2 \)) reads

\[ \frac{\omega}{4} \mathbf{X}' \mathbf{I} = \frac{\lambda_1}{2} \mathbf{XI} + \frac{\lambda_2}{2} \mathbf{RI}. \]  
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Substituting \( \mathbf{X}' = (\mathbf{X}_m + \mathbf{X}_e) / \omega \) from (12) and \( \mathbf{X} = \mathbf{X}_m - \mathbf{X}_e \) from (9) yields

\[ \frac{1}{4} (\mathbf{X}_m + \mathbf{X}_e) \mathbf{I} = \frac{\lambda_1}{2} (\mathbf{X}_m - \mathbf{X}_e) \mathbf{I} + \frac{\lambda_2}{2} \mathbf{RI}, \]  
(45)

\[ (\frac{1}{2} - \lambda_1) \mathbf{X}_m + (\frac{1}{2} + \lambda_1) \mathbf{X}_e \mathbf{I} = \lambda_2 \mathbf{RI}, \]  
(46)

Substituting \( \nu = \frac{1}{2} + \lambda_1 \) finally creates the link between two presented methods

\[ (1 - \nu) \mathbf{X}_m + \nu \mathbf{X}_e \mathbf{I} = \lambda_2 \mathbf{RI}. \]  
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\[ AI = \lambda_1 CI + \lambda_2 BI. \] (43)

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\[ \frac{\omega}{4} X'I = \frac{\lambda_1}{2} XI + \frac{\lambda_2}{2} RI. \] (44)

Substituting \( X' = (X_m + X_e)/\omega \) from (12) and \( X = X_m - X_e \) from (9) yields
\[ \frac{1}{4} (X_m + X_e) I = \frac{\lambda_1}{2} (X_m - X_e) I + \frac{\lambda_2}{2} RI, \] (45)
\[ \left( \left( \frac{1}{2} - \lambda_1 \right) X_m + \left( \frac{1}{2} + \lambda_1 \right) X_e \right) I = \lambda_2 RI, \] (46)
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Substituting \( \nu = 1/2 + \lambda_1 \) finally creates the link between two presented methods

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Summary

Part 1 – Minimization of quality factor $Q$

Representation of continuous int.-dif. operators:

1. make the problem algebraic
   - our choice: piecewise basis functions $\{\psi_n\}$ with $\langle \psi_m, L(\psi_n) \rangle$

2. make the problem feasible
   - our choice: entire domain basis functions $\{I_n\}$ with $\langle I_m, L(I_n) \rangle$

Quality factor $Q$ without external tuning

1st method we saw that the self-resonant current is optimal

2nd method the resonant (or other) constraint can easily be set

3rd method yields the self-resonant current automatically
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Part 2 – Optimal currents

determination of the optimal currents is well-established
more challenging problems can be now studied
all concepts are still half-way to their applicability (feeding)
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Part 2 – Optimal currents

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- more challenging problems can be now studied
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Future work
- arrays/scatterers
- excitation placement, number of feeders
- shape modifications
Today’s presentation will be followed by the second part:

- **Capek, M.:** Implementation of Source Concept in Matlab, (Jan. 19 **Thu 11 AM**).

You will learn about

- implementation of the Source Concept,
- (problematic) feeding of optimal currents,
- Matlab (proc and cons),
- new features in Matlab,
- developing big project in Matlab (how to stay sane).
Questions?

For complete PDF presentation see capek.elmag.org

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