Characteristic Modes

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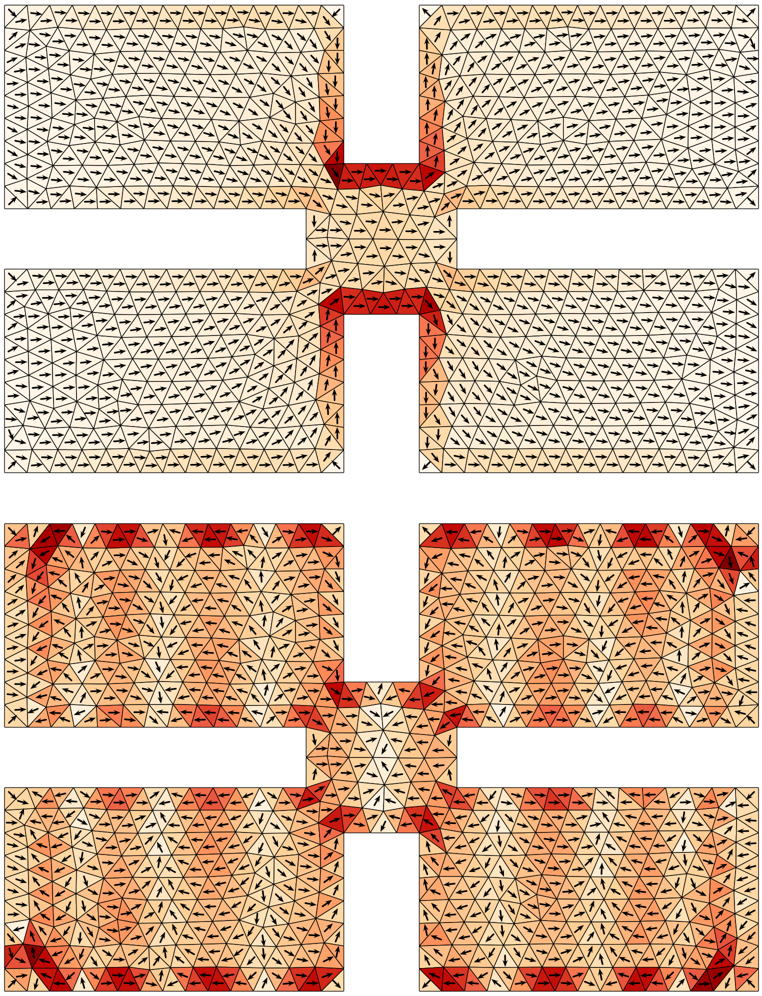

Front page: Sierpinsky fractal of the 1st iteration, made of perfect electric conductor, discretized into 1500 RWG basis functions and decomposed into characteristic modes at $ka = 1.75$. The current density of the dominant (top) and the 112th modes (bottom) are depicted. The 112th mode has been calculated using projection matrix $\mathbf{U}$ to enhance the numerical dynamics since such a high mode cannot be reached by standard procedure.

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Symbols

c_0	speed of light, $c_0 = 299792458$	$\left[\mathrm{m\,s^{-1}}\right]$
μ_0	vacuum permeability, $\mu_0 = 4\pi 10^{-7}$	$\left[\mathrm{F\,m^{-1}}\right]$
ε_0	vacuum permittivity, $\varepsilon_0 = 1/\left(\mu_0 c_0^2\right)$	$\left[\mathrm{H\,m^{-1}}\right]$
Z_0	impedance of free space, $Z_0 = \sqrt{\mu_0/\varepsilon_0}$	$[\Omega]$
σ	conductivity	$\left[\mathrm{S\,m^{-1}}\right]$
$\boldsymbol{\psi}$	basis functions	$[-]$
$\mathit{\Omega}$	radiator's body	$\left[\mathrm{m^2}\right]$
$\mathbf{I}$	vector of expansion coefficients	$\left[\mathrm{Am^{-1}}\right]$

Nomenclature

j	imaginary unit
Re, Im	real, imaginary part
$\boldsymbol{a}$, $\boldsymbol{A}$	vectors
$\hat{\boldsymbol{n}}$	unit vector
$\mathbf{A}$	matrix
$\mathcal{A}$	operator
$\mathbf{A}^{\mathrm{T}}$	matrix transpose
$\mathbf{A}^{\mathrm{H}}$	Hermitian transpose
$\mathbf{A}^{*}$	complex conjugate
$\{a_n\}$	set or collection
$[x_{\min}, x_{\max}]$	interval
$\mathcal{O}$	algorithm complexity
$\mathbb{Z}$, $\mathbb{N}^{+}$, $\mathbb{R}$, $\mathbb{C}$	integer, nonnegative, real, complex numbers

Foreword

Much has been written about theory of CMs, still, some common misunderstandings are persisting even inside the community of CMs. Therefore this text, meant originally as a short contribution to Wikipedia web page (now Chapter 1), is being prepared.

Two additional reasons why to prepare text about CMs can be tracked down. The first one is related to a fact that there is no complete text on this topic, which touches, influences and so changes many of the today's antenna design fields. The topic is only partly covered by classical works [1] and [2], however, many new findings have been realized from that time. There is a novel monograph [3], however, it is composed of a collections of papers only. The second reason is related to the finishing phase of the Antenna Toolbox for MATLAB (AToM) project and to upcomming European School of Antennas series on CMs.

Organization of the Document

The first chapter is a brief introduction into characteristic modes. This text has originally been written for Wikipedia.

Acknowledgment

Chapter 1

Characteristic Modes in a Nutshell

Characteristic modes (CMs) form a set of functions which, under specific boundary conditions, diagonalizes operator relating field and induced sources. The set of CMs is unique and complete (at least theoretically) and thereby capable of describing the behavior of a studied object in full.

This chapter summarizes the basics of CMs decomposition and it have been poster on Wikipedia[1]. Its study will take from 15 to 30 minutes.

1.1 Background

CMs decomposition was originally introduced as set of modes diagonalizing a scattering matrix [4, 5]. The theory has, subsequently, been generalized by Harrington and Mautz for antennas [6, 7]. Harrington, Mautz and their students also successively developed several other extensions of the theory [8, 9, 10, 11]. Even though some precursors were published back in the late 1940s [12, Sec. 9.24], the full potential of CMs has remained unrecognized for an additional 40 years. The capabilities of CMs were revisited [13] in 2007 and, since then, interest in CMs has dramatically increased. The subsequent boom of CMs theory is reflected by the number of prominent publications and applications.

1.2 Definition

For the sake of simplicity, only the original form of the CMs – formulated for perfect electric conductor (PEC) bodies in free space – will be treated in this introductory chapter. The electromagnetic quantities will solely be represented as Fourier's images in frequency domain [14]. Lorenz's gauge [15] is used.

The scattering of an electromagnetic wave on a PEC body is represented via a boundary condition on the PEC body, namely

$$\hat{\boldsymbol{n}} \times \boldsymbol{E}^{\mathrm{i}} = -\hat{\boldsymbol{n}} \times \boldsymbol{E}^{\mathrm{s}} \tag{1.1}$$

[1] Notice that some changes on Wikipedia can appeared since the moderators and other users.

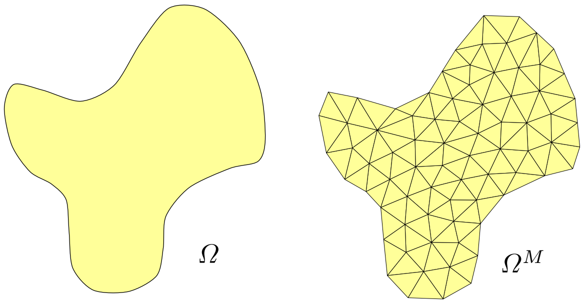

Figure 1.1: Example of a scatterer $\varOmega$ composed of a PEC (left) and its triangular (Delaunay) discretization $\varOmega^M$.

with $\hat{\boldsymbol{n}}$ representing unitary normal to the PEC surface, $\boldsymbol{E}^{\mathrm{i}}$ representing incident electric field intensity, and $\boldsymbol{E}^{\mathrm{s}}$ representing scattered electric field intensity defined as

$$\boldsymbol{E}^{\mathrm{s}} = -\mathrm{j}\omega \boldsymbol{A} - \nabla \varphi, \tag{1.2}$$

with j being imaginary unit, ω being angular frequency, $\boldsymbol{A}$ being vector potential

$$\boldsymbol{A}(\boldsymbol{r}) = \mu_0 \int\limits_{\varOmega} \boldsymbol{J}(\boldsymbol{r}') \, G(\boldsymbol{r}, \boldsymbol{r}') \, \mathrm{d}S, \tag{1.3}$$

μ_0 being vacuum permeability, φ being scalar potential

$$\varphi(\boldsymbol{r}) = -\frac{1}{\mathrm{j}\omega\varepsilon_0} \int\limits_{\varOmega} \nabla \cdot \boldsymbol{J}(\boldsymbol{r}') \, G(\boldsymbol{r}, \boldsymbol{r}') \, \mathrm{d}S, \tag{1.4}$$

ε_0 being vacuum permittivity, $G(\boldsymbol{r}, \boldsymbol{r}')$ being scalar Green's function

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{\mathrm{e}^{-\mathrm{j}k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi |\boldsymbol{r}-\boldsymbol{r}'|} \tag{1.5}$$

and k being wavenumber. The integro-differential operator $\hat{\boldsymbol{n}} \times \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{J})$ is the one to be diagonalized via CMs.

The governing equation of CMs decomposition is

$$\mathcal{X}(\boldsymbol{J}_n) = \lambda_n \mathcal{R}(\boldsymbol{J}_n) \tag{1.6}$$

with $\mathcal{R}$ and $\mathcal{X}$ being real and imaginary parts of impedance operator $\mathcal{Z}$, respectively, defined as

$$\mathcal{Z}(\boldsymbol{J}) = \mathcal{R}(\boldsymbol{J}) + \mathrm{j}\mathcal{X}(\boldsymbol{J}) = \hat{\boldsymbol{n}} \times \hat{\boldsymbol{n}} \times \boldsymbol{E}^{\mathrm{s}}(\boldsymbol{J}). \tag{1.7}$$

The outcome of (1.6) is a set of CMs $\{\boldsymbol{J}_n\}$, $n \in \{1, 2, \dots\}$ accompanied by associated characteristic numbers $\{\lambda_n\}$. Clearly, (1.6) is a generalized eigenvalue problem (GEP), which, however, cannot be analytically solved (except for a few canonical bodies [16]). Therefore, the numerical solution described in the following paragraph is commonly employed.

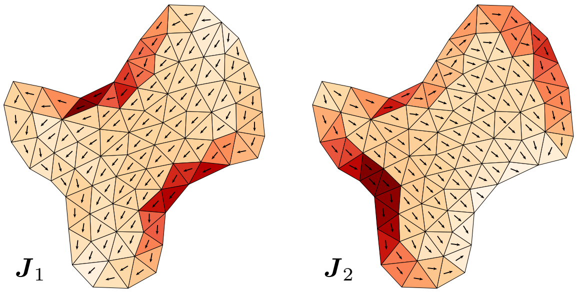

Figure 1.2: The first two CMs for a structure from Fig. 1.1.

1.3 Matrix formulation

Discretization $\mathcal{D}$ of the body of the scatterer Ω into M subdomains as $\Omega^M = \mathcal{D}(\Omega)$ and using a set of linearly independent piece-wise continuous functions $\{\boldsymbol{\psi}_n\}$, $n \in \{1, \ldots, N\}$, allows current density $\boldsymbol{J}$ to be represented as

$$\boldsymbol{J}(\boldsymbol{r}) \approx \sum_{n=1}^{N} I_n \boldsymbol{\psi}_n(\boldsymbol{r}) \tag{1.8}$$

and by applying the Galerking method, the impedance operator (1.7)

$$\mathbf{Z} = \mathbf{R} + \mathrm{j}\mathbf{X} = [Z_{uv}] = \left[\int_{\Omega} \boldsymbol{\psi}_u^* \cdot \mathcal{Z}(\boldsymbol{\psi}_v) \, \mathrm{d}S \right]. \tag{1.9}$$

The eigenvalue problem (1.6) is then recast into its matrix form

$$\mathbf{X}\mathbf{I}_n = \lambda_n \mathbf{R}\mathbf{I}_n, \tag{1.10}$$

which can easily be solved using, *e.g.*, the generalized Schur decomposition [17] or the implicitly restarted Arnoldi method [18] yielding a finite set of expansion coefficients $\{\mathbf{I}_n\}$ and associated characteristic numbers $\{\lambda_n\}$. The properties of CMs decomposition are investigated below.

1.4 Properties

The properties of CMs decomposition are demonstrated in its matrix form.

First, recall that the bilinear forms

$$P_\mathrm{r} \approx \frac{1}{2}\mathbf{I}^\mathrm{H}\mathbf{R}\mathbf{I} \geq 0 \tag{1.11}$$

and

$$2\omega \left(W_\mathrm{m} - W_\mathrm{e}\right) \approx \frac{1}{2}\mathbf{I}^\mathrm{H}\mathbf{X}\mathbf{I}, \tag{1.12}$$

where superscript $^\mathrm{H}$ denotes the Hermitian transpose and where $\mathbf{I}$ represents an arbitrary surface current distribution, correspond to the radiated power and the reactive net power [19], respectively. The following properties can then be easily distilled:

- The weighting matrix $\mathbf{R}$ is theoretically positive definite and $\mathbf{X}$ is indefinite. The Rayleigh quotient
$$\lambda_n \approx \frac{\mathbf{I}_n^{\mathrm{H}}\mathbf{X}\mathbf{I}_n}{\mathbf{I}_n^{\mathrm{H}}\mathbf{R}\mathbf{I}_n} \tag{1.13}$$
then spans the range of $-\infty \leq \lambda_n \leq \infty$ and indicates whether the CM is capacitive ($\lambda_n < 0$), inductive ($\lambda_n > 0$), or in resonance ($\lambda_n = 0$). In reality, the Rayleigh quotient is limited by the numerical dynamics of the machine precision used and the number of correctly found modes is limited.
- CMs evolve with frequency, *i.e.*, $\lambda_n = \lambda_n(\omega)$, they can cross each other, or they can be the same (in case of degeneracies [20]). For this reason, the tracking of modes is often applied in order to get smooth curves $\lambda_n(\omega)$ [21, 22, 23, 24, 25]. Unfortunately, this process is partly heuristic and the tracking algorithms are still far from perfection [16].
- CMs can be chosen as real-valued functions, $\mathbf{I}_n \in \mathbb{R}^{N\times 1}$. In other words, CMs form a set of equiphase currents.
- CMs decomposition is invariant with respect to the amplitude of CMs. This fact is used to normalize the current so that they radiate unitary radiated power
$$\frac{1}{2}\mathbf{I}_m^{\mathrm{H}}\mathbf{Z}\mathbf{I}_n \approx (1 + \mathrm{j}\lambda_n)\,\delta_{mn}. \tag{1.14}$$
This last relation presents the ability of CMs to diagonalize the impedance operator (1.7) and demonstrates far field orthogonality, *i.e.*,
$$\frac{1}{2Z_0}\int_0^{2\pi}\int_0^{\pi} \mathbf{F}_m^* \cdot \mathbf{F}_n \sin\vartheta \,\mathrm{d}\vartheta\,\mathrm{d}\varphi = \delta_{mn}. \tag{1.15}$$

1.5 Modal quantities

The modal currents can be used to evaluate antenna parameters in their modal form, for example:

- modal far-field $\mathbf{F}_n(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}})$, $\hat{\boldsymbol{e}}$ – polarization, $\hat{\boldsymbol{r}}$ – direction, [6],
- modal directivity $D_n(\hat{\boldsymbol{e}}, \hat{\boldsymbol{r}})$,
- modal radiation efficiency η_n [26],
- modal quality factor Q_n [27],
- modal impedance Z_n.

These quantities can be utilized for analysis, feeding synthesis, radiator's shape optimization, or antenna characterization.

1.6 Applications and Further Development

The number of potential applications is enormous and still growing:

- antenna analysis and synthesis [28, 29, 30],
- design of Multiple-input multiple-output (MIMO) antennas [31, 32, 33, 34, 35],
- compact antenna design of Radio Frequency Identification (RFID) [36, 37],
- Unmanned aerial vehicle (UAV) antennas [38],
- selective excitation of chassis and platforms [39],
- model order reduction [40],
- bandwidth enhancement [41, 42],
- nanotubes [43] and metamaterials [44, 45],
- validation of codes [16].

The prospective topics include

- electrically large structures calculated using Multilevel Fast Multipole Algorithm (MLFMA) [46],
- dielectrics [10, 47],
- utilization of Combined Field Integral Equation [48],
- periodic structures,
- formulation for arrays [**?**].

1.7 Software

CMs decomposition has recently been implemented in major electromagnetic simulators, namely in FEKO [49], CST-MWS [50], and WIPL-D [51]. Other packages are about to support it soon, for example HFSS [52] and CEM One [53]. In addition, there is a plethora of in-house and academic packages which are capable of evaluating CMs and many associated parameters.

1.8 Alternative bases

CMs are useful to understand radiator's operation better. They have been used with great success for many practical purposes. However, it is important to stress that they are not perfect and ti is often better to use other formulations such as energy modes [54], radiation modes [54], stored energy modes [40] or radiation efficiency modes [55].

Chapter 2

Scheduled Content

2.1 History

The history of CMs dates back into the 1940s. Montgomery *et al.* introduced the elements of CMs decomposition for scattering [12]. It took more than 15 years till Garbacz noticed the existence of the theory and, acknowledging the Montgomery's previous work XXX, he significantly elaborated the analytical parts of the theory and provided the exact results for some canonical scatterers XXX. For example, CMs of a spherical shell are extensively used through this work. Garbacz with his colleagues stayed active in the field for many following years, xxxx, establishing thus strong tradition of CMs at the Ohio State University.

Harrington and Mautz recognized the power of CMs for antennas. Therefore, they reformulated the existent theory for antennas utilizing impedance matrix. It was Harrington, who first denoted these modes as *characteristic modes* since "PHRASE". This terms has been readily accepted by the community.

The work on CMs perfectly fits into his other scientific activities, covering wide area of integral equations, antenna synthesis and numerical methods. As will be shown in this text, all these branches are, in fact, related to CMs.

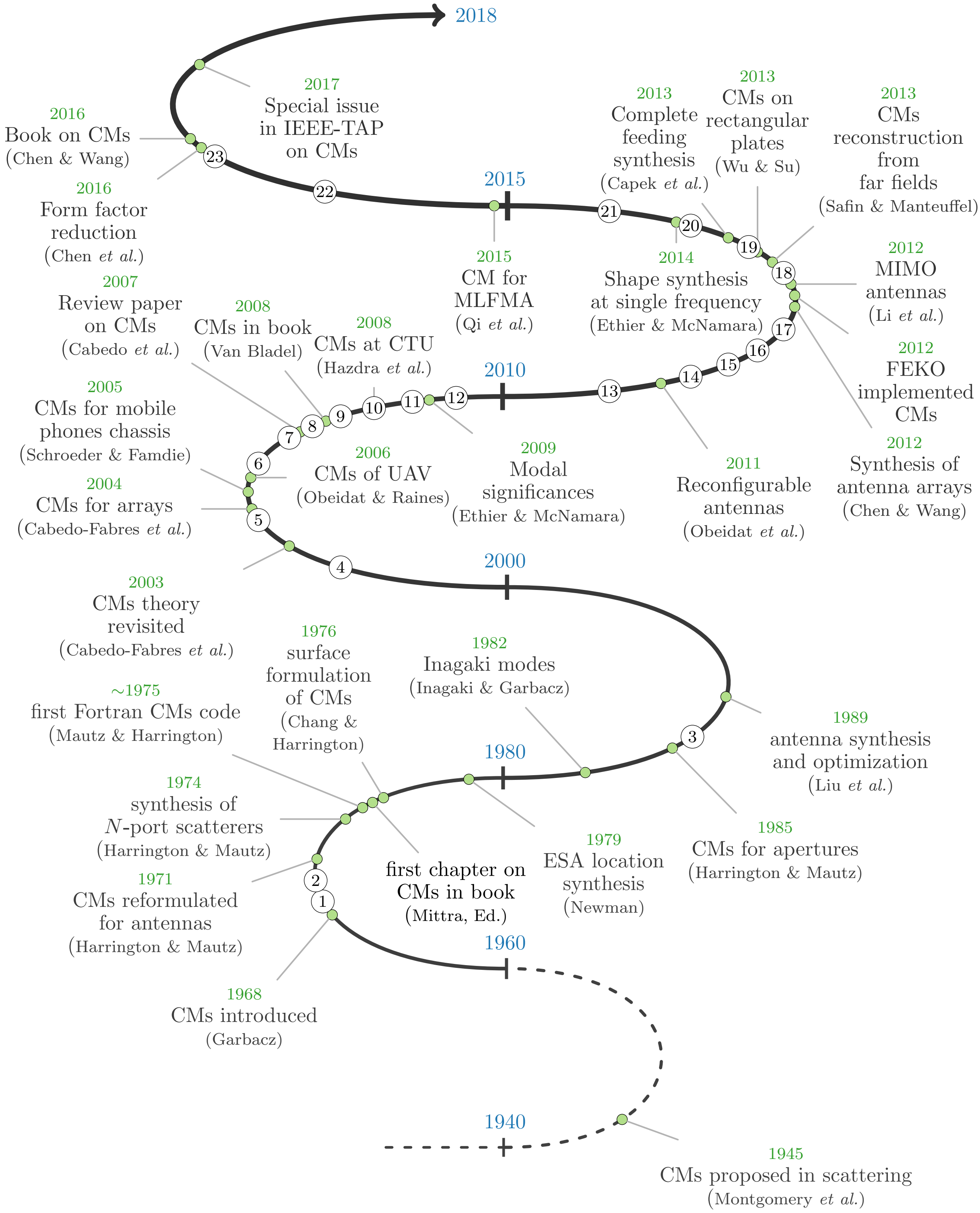

Figure 2.1: Historical overview of CMs development.

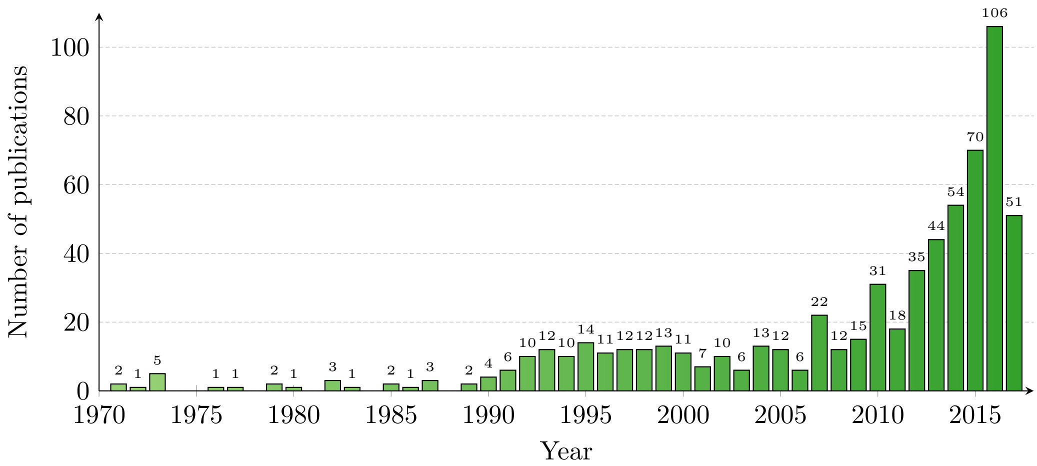

Figure 2.2: Number of scientific papers published both in journals and conference proceedings related to CMs. Data have been exported from Web of Science, terms "characteristic modes" and "characteristic modes decomposition" have been searched.

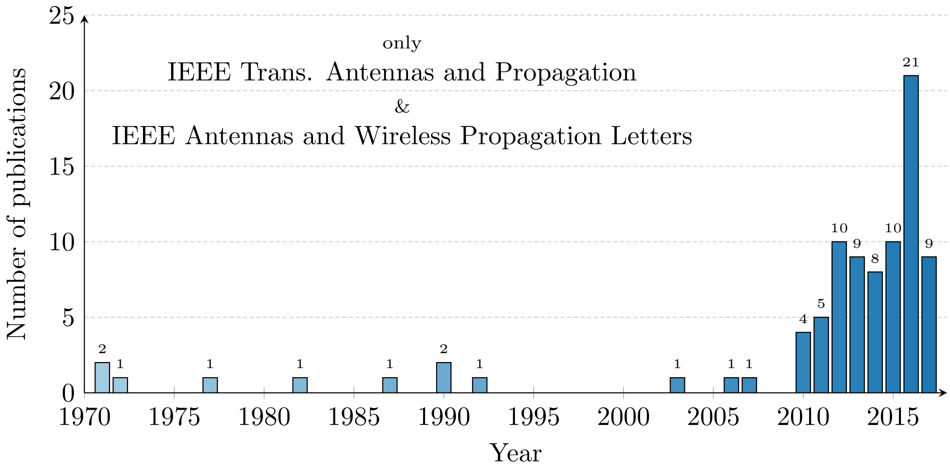

Figure 2.3: Number of scientific papers published in IEEE Trans. AP and IEEE AWPL related to CMs. Data have been exported from Web of Science, terms "characteristic modes" and "characteristic modes decomposition" have been searched.

Table 2.1: Doctoral theses published on topics related to CMs. *Do you know about any other work on the topic? Let me know!*

Year	Author	University	Ref.
1968	Garbacz	Ohio State Uni.	[5]
1969	Turpin	Ohio State Uni.	[56]
1986	Liu	Ohio State Uni.	[57]
2002	Strohschein	Uni. of New Hampshire	[58]
2004	Bekers	Technische Uni. Eindhoven	[59]
2006	Surittikul,	Ohio State Uni.	[60]
2007	Cabedo Fabres	Polytechnic Uni. of Valencia	[61]
2007	Famdie	Uni. of Duisburg-Essen	[62]
2008	Antonino Daviu	Polytechnic Uni. of Valencia	[63]
2009	Hazdra	Czech Technical Uni.	[64]
2009	Sonkki	Uni. of Oulu	[65]
2010	Obeidat	Ohio State Uni.	[66]
2010	Lawrence	Clemson Uni.	[67]
2011	Adams	Univ. of Illinois	[68]
2011	Raines	Ohio State Uni.	[69]
2011	Strojny	Ohio State Uni.	[70]
2012	Ethier	Uni. of Ottawa	[71]
2013	Alroughani	Uni. of Ottawa	[72]
2014	Eichler	Czech Technical Uni.	[73]
2014	Capek	Czech Technical Uni.	[74]
2015	King	Univ. of Illinois	[75]
2015	Rabah	Univ. of Lille 1	[76]
2016	Martens	Uni. of Kiel	[77]
2016	Kundu	North Carolina State Uni.	[78]
2016	Miers	Lund Uni.	[79]
2016	Schab	Uni. of Illinois	[80]
2017	Safin	Uni. of Kiel	[81]

Table 2.2: Selected sessions related to CMs, organized at major events in recent years.

Year	Session	Event
2012	Small Antennas (CA17)	EuCAP
2014	Theory of Characteristic Modes for Antenna System Design in Wireless Communications	APS/URSI
2015	Mode-based strategy for antenna analysis and design Antennas/Multi Applications (C35)	EuCAP
	Modal Analysis of Electromagnetic Structures	EuMC
2016	Theory and Application of Characteristic Modes (CS38a+CS38b)	EuCAP
	Characteristic Mode Analysis for Small Antenna Design	ISAP
	Antenna Design/Analysis Based on Stored Energy and Characteristic Modes	APS/URSI
2017	Characteristic Mode Analysis for Platform-Mounted Antenna Design (CS10)	EuCAP
	New Trends in Characteristic Modes Research (CS30)	EuCAP
	Theory of Characteristic Mode and Its Applications	APCAP
2018	Advances in Theory and Computation of Characteristic Modes	EuCAP
	Advances in Antenna Design and Analysis Using Characteristic Modes	EuCAP

2.1.1 Precursors (1920s – 1960s)

2.1.2 Pioneering Works (1960s – 1970s)

2.1.3 Advancing Theory (1970s – 1990s)

2.1.4 New Era Comming With Powerful Computers (2000s – 2010s)

2.1.5 Not Only Characteristic Modes (2010s)

2.2 Matrix formulation

2.3 Tracking

2.4 Related Modal Metrics

2.5 Modal Antenna Quantities

2.6 Matching Modes With External World

2.7 Application of CMs

2.8 Alternative Bases

2.9 Unsolved problems

1. Completeness and uniqueness of CMs.
2. Numerical dynamics and preconditioning of CMs.
3. Formulation of CMs and existence of a solution for dielectrics.
4. Formulation of CMs and existence of a solution for scatterers with ohmic losses.
5. Tracking of CMs, crossing avoidances and their interpretation.
6. CMs for electrically large and multi-scale objects (including computational and application aspects).
7. Feeding synthesis using CMs (is it even possible, or at least effective?).

Interesting problems to be solved:

1. All the above.
2. Adaptation of theoretical apparatus about operators from QET to CMs.
3. Porting CMs theory to close branches (what is their meaning in acoustics, etc.).
4. Optimal compositions of modes for various antenna parameters (partly done).
5. Finish list of model order reduction techniques with direct application of CMs.

2.10 Criticism and Deficiencies

Feeding – not designed for these purposes

Numerical dynamics – only first couple (tens) of modes

Orthogonality – problems with losses, dielectrics

Uniqueness – problem with volumetric structures, non-radiating currents

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