Fundamental Bounds In Electromagnetism

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1. Optimal Design and Its Feasibility
2. Fundamental Bounds
3. First Attempts
4. Example: Bounds on Radiation Efficiency
5. Utilizing Integral Equations
6. Solution to QCQP Problems
7. Tightness of the Bounds

Electrically small antenna inside a circumscribing sphere of a radius $a$. 

Document available at capek.elmag.org.

To see the graphics in motion, open this document in Adobe Reader! 
Designing EM Devices...
What is the optimal design?
Designing EM Devices...

What is the optimal design?
Optimal design for what...?
Designing EM Devices...

What is the optimal design?
Optimal design for what...?
What is the optimal performance?

Folded loop (handsets)  E-shaped patch (GPS, WLAN)  “Mag. monopoles” (PGB, HIS)  Meandered dipole (RFID)  Monopoles/PIFAs (LTE)

connected devices (billions)


0 20 40 60 80

2015
2016
2017
2018
2019
2020
2021
2022
2023
2024
2025
0
20
40
60
80

Connected devices (billions)
Degrees of Freedom and Figure of Merits

- Materials
- Performance
- Electrical size
- Geometry
- Excitation
- Regularity
- Features

Design domain: Analysis
Synthesis: Criteria domain
Optimal Design and Its Feasibility

Analysis

► Shape is given, feeding is known.
► The task is to determine EM quantities.
Optimal Design and Its Feasibility

Analysis

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Synthesis (Inverse design)

- EM behavior is specified.
- The task is to find optimal shape.
### Analysis
- Shape is given, feeding is known.
- The task is to determine EM quantities.
- Mastered.
- Plenty of circuit & full-wave EM simulators.

### Synthesis (Inverse design)
- EM behavior is specified.
- The task is to find optimal shape.
Optimal Design and Its Feasibility

Analysis
- Shape is given, feeding is known.
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- Mastered.
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Synthesis (Inverse design)
- EM behavior is specified.
- The task is to find optimal shape.
- Unsolved (except of rare cases).
- NP-hard/NP-complete.
Design Strategies

1. Designer’s skill, experiences, and intuition.
2. Parameter sweep for predefined shapes.
3. Design libraries.
4. Local optimization (gradient-based).
5. Global optimization (heuristics).
6. Memetics, machine-learning-assisted techniques.

\[ i = 0, \frac{Q}{Q_{lb}^{TM}} = 119.35 \]
Design Curve
Design Curve

Parameter sweep

Empirical design

Time to design a device:
- seconds
- minutes
- hours
- days
- weeks

Performance
Design Curve

- Optimal Design and Its Feasibility
- Design Curve

- Parameter sweep
- Empirical design
- Topology optimization
- Heuristic algorithm

- Time to design a device:
  - Seconds
  - Minutes
  - Hours
  - Days
  - Weeks

- Performance
Design Curve

The graph illustrates the relationship between the time to design a device and its performance/bound. It shows the progress through different design phases:

- **Empirical Design**: Initial stage with limited data.
- **Parameter Sweep Optimization**: Improving performance with systematic adjustments.
- **Topology Optimization**: Refining design for better performance.
- **Heuristic Algorithm**: Advanced computational methods for optimization.

The graph is divided into time intervals:

- **Seconds**
- **Minutes**
- **Hours**
- **Days**
- **Weeks**

The fundamental bound is represented by a horizontal line at the top of the graph, indicating the maximum achievable performance.
Example: Energy Extraction

Combustion
Fundamental Bounds

Example: Energy Extraction

\[ W_{\text{bound}} \approx 10^{-9} \]

\[ W_{\text{bound}} \approx 10^{-3} \]

\[ W_{\text{bound}} = 1 \]

Combustion

Nuclear fission
Fundamental Bounds

Example: Energy Extraction

Combustion
Nuclear fission
Nuclear fusion

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What is the physical bound on energy production from fuel with mass \( m \)?

\[ W_{\text{bound}} = mc^2 \]
Fundamental Bounds

Example: Energy Extraction

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\frac{W}{W_{\text{bound}}} \approx 10^{-9}
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Combustion  Nuclear fission  Nuclear fusion

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Nuclear fission

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Nuclear fusion

\[
\frac{W}{W_{\text{bound}}} = 1
\]

Annihilation of matter and antimatter

What is the physical bound on energy production from fuel with mass \( m \)?

\[ W_{\text{bound}} = mc^2 \]
Approaching Fundamental Bounds in EM – Overview

- **Circuit quantities** (e.g., equivalent circuits).
  - Wheeler (radiation power factor, 1947)
  - Chu (Q-factor, 1948)
  - Fano (matching, 1950)
  - Thal (Q-factor, 1978)
  - Pfeiffer (radiation efficiency, 2017)

![Circuit Diagram]

\[ \varepsilon_0 a \quad \mu_0 a \quad Z_0 \]
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- **Field quantities** (*e.g.*, spherical harmonics).
  - Harrington (gain, 1965)
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  - Gustafsson et al. (2010+)
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- **Related bounds**
  - Shannon (capacity, 1948)
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First Attempts: Directivity

What is the highest achievable directivity of an antenna?
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- It is possible to design an antenna of arbitrarily small dimensions with a directivity as high as desired\(^1\).

First Attempts: Q-factor

What is the highest achievable fractional bandwidth of a single-resonant antenna?
First Attempts: Q-factor

What is the highest achievable fractional bandwidth\(^2\) of a single-resonant antenna?

\[
FBW < \frac{2 |\Gamma|}{Q_{\text{Chu}}} \quad (1)
\]

\[
Q_{\text{Chu}} = \frac{1}{2} \left( \frac{1}{(ka)^3} + \frac{2}{ka} \right) \quad (2)
\]

Key ingredient: Expansion of field into spherical waves.

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First Attempts: Away From Spheres

- Spherical waves are only suitable for spherical design regions.
- The developed bounds are relatively loose as compared to common antenna designs.
First Attempts: Away From Spheres

- Spherical waves are only suitable for spherical design regions.
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“Shape-specific” fundamental bounds\(^3\)

Given a specific design region, what is the best performance we can get from a device built in this region from a given material?

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Example: Radiation Efficiency and Dissipation Factor

Radiation efficiency\(^4\):

\[
\eta_{\text{rad}} = \frac{P_{\text{rad}}}{P_{\text{rad}} + P_{\text{lost}}} = \frac{1}{1 + \delta_{\text{lost}}}
\]  \hspace{1cm} (3)

Dissipation factor\(^5\) \(\delta\):

\[
\delta_{\text{lost}} = \frac{P_{\text{lost}}}{P_{\text{rad}}}
\]  \hspace{1cm} (4)

\(\triangleright\) fraction of quadratic forms (can be scaled with resistivity model).

---

Example: Radiation Efficiency and Dissipation Factor

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---


Utilizing Integral Equations

Integral Operators and Their Algebraic Representation

Radiated and reactive power:

\[ P_{\text{rad}} + 2j\omega (W_m - W_e) = \frac{1}{2} \langle J (r), Z [J (r)] \rangle \]

Lost power (surface resistivity model):

\[ P_{\text{lost}} = \frac{1}{2} \langle J (r), \Re \{ Z_s \} J (r) \rangle \]

The same approach as with the method of moments (MoM)

\[ J (r) \approx \sum_n I_n \psi_n (r) \]
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Algebraic Representation of Integral Operators
Radiated and reactive power

\[ P_{\text{rad}} + 2j \omega (W_m - W_e) = \frac{1}{2} \langle \mathbf{J} (r) , \mathbf{Z} [\mathbf{J} (r)] \rangle \approx \frac{1}{2} \mathbf{I}^H \mathbf{Z} \mathbf{I} \]  

Electric Field Integral Equation\(^7\) (EFIE), \( \mathbf{Z} = [Z_{mn}] \):

\[ Z_{mn} = \int_{\Omega} \psi_m \cdot \mathbf{Z} (\psi_n) \, dS = jkZ_0 \int_{\Omega} \int_{\Omega} \psi_m (r_1) \cdot \mathbf{G} (r_1, r_2) \cdot \psi_n (r_2) \, dS_1 \, dS_2. \]  

- Dense, symmetric matrix.
- An output from PEC 2D/3D MoM code (Ansys FEKO, CST MWS, HFSS, ...).
Algebraic Representation of Integral Operators

Radiated and reactive power

\begin{equation}
    P_{\text{rad}} + 2j\omega (W_m - W_e) = \frac{1}{2} \langle \mathbf{J}(r), \mathbf{Z} [\mathbf{J}(r)] \rangle \approx \frac{1}{2} \mathbf{I}^H \mathbf{Z} \mathbf{I}
\end{equation}

Electric Field Integral Equation \(^7\) (EFIE), \( \mathbf{Z} = [Z_{mn}] \):

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---

Algebraic Representation of Integral Operators

Radiated and reactive power

\[ P_{\text{rad}} + 2j\omega (W_m - W_e) = \frac{1}{2} \langle J(r), Z[J(r)] \rangle \approx \frac{1}{2} I^H Z I \]  \hspace{1cm} (5)

Electric Field Integral Equation\(^7\) (EFIE), \(Z = [Z_{mn}]\):

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Algebraic Representation of Integral Operators

Lost power

\[ P_{\text{lost}} = \frac{1}{2} \langle J(r), \text{Re} \{ Z_s \} [J(r)] \rangle \] (7)
Algebraic Representation of Integral Operators

Lost power

\[
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\]  \hspace{1cm} (7)

\[
L_{mn} = \int_{\Omega} \psi_m \cdot \psi_n \, dS
\]  \hspace{1cm} (8)

Surface resistivity model:

\[
Z_s = \frac{1 + j}{\sigma \delta}
\]  \hspace{1cm} (9)

with skin depth \( \delta = \sqrt{2/\omega \mu_0 \sigma} \).
Algebraic Representation of Integral Operators

Lost power

\[ P_{\text{lost}} = \frac{1}{2} \langle \mathbf{J}(\mathbf{r}), \text{Re} \{ Z_s \} [\mathbf{J}(\mathbf{r})] \rangle \approx \frac{1}{2} \mathbf{I}^H \mathbf{L} \mathbf{I} \] (7)

\[ L_{mn} = \int_{\Omega} \mathbf{\psi}_m \cdot \mathbf{\psi}_n \, dS \] (8)

Surface resistivity model:

\[ Z_s = \frac{1+j}{\sigma \delta} \] (9)

with skin depth \( \delta = \sqrt{2/\omega \mu_0 \sigma} \).

- Sparse matrix (diagonal for non-overlapping functions \( \{ \mathbf{\psi}_m(\mathbf{r}) \} \)).
- The entries \( L_{mn} \) are known analytically.
A Note: MoM Solution × Current Impressed in Vacuum

Solution to $\mathbf{I} = \mathbf{Z}^{-1}\mathbf{V}$ for an incident plane wave.

A current can be chosen completely freely, only the excitation $\mathbf{V} = \mathbf{Z}\mathbf{I}$ may not be realizable.
A Note: MoM Solution × Current Impressed in Vacuum

MoM solution

Solution to $I = Z^{-1}V$ for an incident plane wave.

Current impressed in vacuum

Solution to $XI_i = \lambda_i R I_i$ (the first inductive mode).

A current can be chosen completely freely, only the excitation $V = ZI$ may not be realizable.
Utilizing Integral Equations

Fundamental Bounds as QCQP Problems

- Having quadratic forms for the physical quantities, the antenna metrics may be optimized.

<table>
<thead>
<tr>
<th>Maximum radiation efficiency</th>
<th>Maximum self-resonant radiation efficiency</th>
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<tbody>
<tr>
<td>Problem $\mathcal{P}_1$:</td>
<td>Problem $\mathcal{P}_2$:</td>
</tr>
<tr>
<td>minimize $P_{loss}$</td>
<td>minimize $P_{loss}$</td>
</tr>
<tr>
<td>subject to $P_{rad} = 1$</td>
<td>subject to $P_{rad} = 1$</td>
</tr>
<tr>
<td></td>
<td>$\omega (W_m - W_e) = 0$</td>
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Utilizing Integral Equations

Fundamental Bounds as QCQP Problems

- Having quadratic forms for the physical quantities, the antenna metrics may be optimized.
- The problems $\mathcal{P}_1$ and $\mathcal{P}_2$ are quadratically constrained quadratic programs (QCQP).

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</table>
| subject to $\mathbf{I}^H\mathbf{R}\mathbf{I} = 1$ | subject to $\mathbf{I}^H\mathbf{R}\mathbf{I} = 1$  
|                             | $\mathbf{I}^H\mathbf{X}\mathbf{I} = 0$  |

---

Solution to Radiation Efficiency Bound ($\mathcal{P}_1$)

Lagrangian reads

$$\mathcal{L}(\lambda, \mathbf{I}) = \mathbf{I}^H \mathbf{L} \mathbf{I} - \lambda \left( \mathbf{I}^H \mathbf{R} \mathbf{I} - 1 \right).$$  \hspace{1cm} (10)

Stationary points

$$\frac{\partial \mathcal{L}}{\partial \mathbf{I}^H} = \mathbf{L} \mathbf{I} - \lambda \mathbf{R} \mathbf{I} = 0$$  \hspace{1cm} (11)

are solution to generalized eigenvalue problem (GEP):

$$\mathbf{L} \mathbf{I}_i = \lambda_i \mathbf{R} \mathbf{I}_i.$$  \hspace{1cm} (12)

Substituting a discrete set of stationary points $\{\mathbf{I}_i, \lambda_i\}$ back to (10) and minimizing gives

$$\min_{\{\mathbf{I}_i\}} \mathcal{L}(\lambda, \mathbf{I}) = \lambda_1.$$  \hspace{1cm} (13)
Solution to Radiation Efficiency Bound \((\mathcal{P}_1)\)

Lagrangian reads

\[
\mathcal{L}(\lambda, \mathbf{I}) = \mathbf{I}^H \mathbf{LI} - \lambda (\mathbf{I}^H \mathbf{RI} - 1).
\]  \(\text{(10)}\)

Stationary points

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\]  \(\text{(13)}\)
Solution to Radiation Efficiency Bound ($\mathcal{P}_1$)

Lagrangian reads
\[
\mathcal{L} (\lambda, I) = I^H L I - \lambda (I^H R I - 1). \tag{10}
\]

Stationary points
\[
\frac{\partial \mathcal{L}}{\partial I^H} = LI - \lambda RI = 0 \tag{11}
\]
are solution to generalized eigenvalue problem (GEP):
\[
LI_i = \lambda_i RI_i. \tag{12}
\]

Substituting a discrete set of stationary points \(\{I_i, \lambda_i\}\) back to (10) and minimizing gives
\[
\min_{\{I_i\}} \mathcal{L} (\lambda, I) = \lambda_1. \tag{13}
\]
Example: Radiation Efficiency Bound of an L-plate ($\mathcal{P}_1$)

$ka = 1, \quad R_s = 0.01 \, \Omega/\square.$

Optimal current (1st mode), $Z_0/R_s (ka)^2 \delta_{\text{loss}} = 17.6.$
**Example:** Radiation Efficiency Bound of an L-plate ($\mathcal{P}_1$)

\[ ka = 1, \; R_s = 0.01 \, \Omega/\square. \]

Optimal current (1st mode), \( Z_0/R_s (ka)^2 \delta_{\text{loss}} = 17.6. \)

The 2nd current mode, \( Z_0/R_s (ka)^2 \delta_{\text{loss}} = 19.2. \)

- Constant current has the lowest ohmic losses compared to its radiation.
Example: Radiation Efficiency Bound of an L-plate ($\mathcal{P}_1$)

$ka = 1$, $R_s = 0.01 \Omega/\square$.

Optimal current (1st mode), $Z_0/R_s (ka)^2 \delta_{\text{loss}} = 17.6$.

The 2nd current mode, $Z_0/R_s (ka)^2 \delta_{\text{loss}} = 19.2$.

- Constant current has the lowest ohmic losses compared to its radiation.
- Clearly, such current is not realizable (and singular on the boundary).
Solution to Self-Resonant Radiation Efficiency Bound ($\mathcal{P}_2$)

The same solving procedure\(^9\) as with problem $\mathcal{P}_1$, two Lagrange multipliers, however:

$$\mathcal{L} (\lambda_1, \lambda_2, \mathbf{I}) = \mathbf{I}^H \mathbf{L} \mathbf{I} - \lambda_1 (\mathbf{I}^H \mathbf{R} \mathbf{I} - 1) - \lambda_2 \mathbf{I}^H \mathbf{X} \mathbf{I}. \tag{14}$$

Stationary points

$$(\mathbf{L} - \lambda_2 \mathbf{X}) \mathbf{I}_i = \lambda_1, \mathbf{R} \mathbf{I}_i. \tag{15}$$

Solution to Self-Resonant Radiation Efficiency Bound ($\mathcal{P}_2$)

The same solving procedure\(^9\) as with problem $\mathcal{P}_1$, two Lagrange multipliers, however:

$$\mathcal{L} (\lambda_1, \lambda_2, I) = I^H LI - \lambda_1 (I^H RI - 1) - \lambda_2 I^H XI. \quad (14)$$

Stationary points

$$(L - \lambda_2 X) I_i = \lambda_{1,i} RI_i. \quad (15)$$

---

Example: Optimal Currents for L-Shape Plate ($\mathcal{P}_1$ & $\mathcal{P}_2$)

$k a = 1$, $R_s = 0.01 \Omega / \square$.

Optimal current for $\mathcal{P}_1$, $Z_0/R_s (k a)^2 \delta_{\text{loss}} = 17.6$.

Optimal current for $\mathcal{P}_2$, $Z_0/R_s (k a)^4 \delta_{\text{loss}} = 52.3$. 
Example: Optimal Currents for L-Shape Plate ($\mathcal{P}_1$ & $\mathcal{P}_2$)

$k a = 1$, $R_s = 0.01 \Omega/\square$.

Optimal current for $\mathcal{P}_1$,
$$Z_0/R_s (ka)^2 \delta_{\text{loss}} = 17.6.$$  

Optimal current for $\mathcal{P}_2$,
$$Z_0/R_s (ka)^4 \delta_{\text{loss}} = 52.3.$$  

The same approach may be applied for any representation of the integral operators.

- Surface MoM, separable bodies, volumetric MoM, hybrid integral methods.
Trade-off Between Antenna Metrics

Example: Radiation efficiency vs. antenna bandwidth\(^{10}\), \(ka = 1/2\), \(R_s = 1 \Omega/\Box\)

\[ Q_{\text{rad}} = \frac{Q}{\eta} \]

\[ \delta/R_s \]

\[ Q/\eta \]

\[ \text{ext. tuned} \]

\[ \text{self-resonant} \]

---

TARC Minimization

Total active reflection coefficient (TARC)

\[
\Gamma^t = \sqrt{1 - \frac{P_{\text{rad}}}{P_{\text{in}}}} = \sqrt{1 - \frac{v^H g_0 v}{v^H k_i^H k_i v}} \tag{16}
\]

is to be minimized with QCQP\textsuperscript{11}:

maximize \quad v^H g_0 v

subject to \quad v^H k_i^H k_i v = 1 \tag{17}

Solution to QCQP Problems

TARC Minimization

Total active reflection coefficient (TARC)

\[ \Gamma^t = \sqrt{1 - \frac{P_{\text{rad}}}{P_{\text{in}}}} = \sqrt{1 - \frac{\mathbf{v}^H \mathbf{g}_0 \mathbf{v}}{\mathbf{v}^H \mathbf{k}_i^H \mathbf{k}_i \mathbf{v}}} \] (16)

is to be minimized with QCQP\textsuperscript{11}:

\[
\begin{align*}
\text{maximize} & \quad \mathbf{v}^H \mathbf{g}_0 \mathbf{v} \\
\text{subject to} & \quad \mathbf{v}^H \mathbf{k}_i^H \mathbf{k}_i \mathbf{v} = 1
\end{align*}
\] (17)

Various levels of complexity:

- optimal excitation of ports,
- optimal placement of ports,

TARC Minimization

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\end{align*}

(17)

Various levels of complexity:

- optimal excitation of ports,
- optimal placement of ports,
- optimal number of ports,
- optimal matching circuitry.

\textsuperscript{11}M. Capek, L. Jelinek, and M. Masek, “Finding optimal total active reflection coefficient and realized gain for multi-port lossy antennas,” IEEE Transactions on Antennas and Propagation, 2021, early access
Shapes Known to Be Optimal (In Certain Sense)

Radiation Q-factor

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Q-factor of meanderline antennas compared to the bound.

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Cloaking efficiency (extinction cross section)

A (fixed) rod over a slab (optimized).
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Cloaking efficiency (extinction cross section)

A (fixed) rod over a slab (optimized).

Cloaking efficiency of optimized slabs compared to the bound $\eta_{\text{ub cloak}}$. 

$$ε_{r,c} = -10 - j1$$

$$ε_{r,c} = -10 - j0.01$$

$k a_u$
Conclusion

Bounds (QCQP)

- Help us to understand principal limits.
- We know when to stop with the design procedure.
- Applicable to arbitrarily shaped bodies.
- Inhomogeneous materials, combined metrics, trade-offs.
- Supports constraints on input impedance, complex power, directional constraints, polarization, etc.
- Sometimes directly realizable (port-modes).

Future

- Other metrics and their bounds.
- So far only single-frequency.
- Piecewise constraints (local power conservation).
Questions?

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